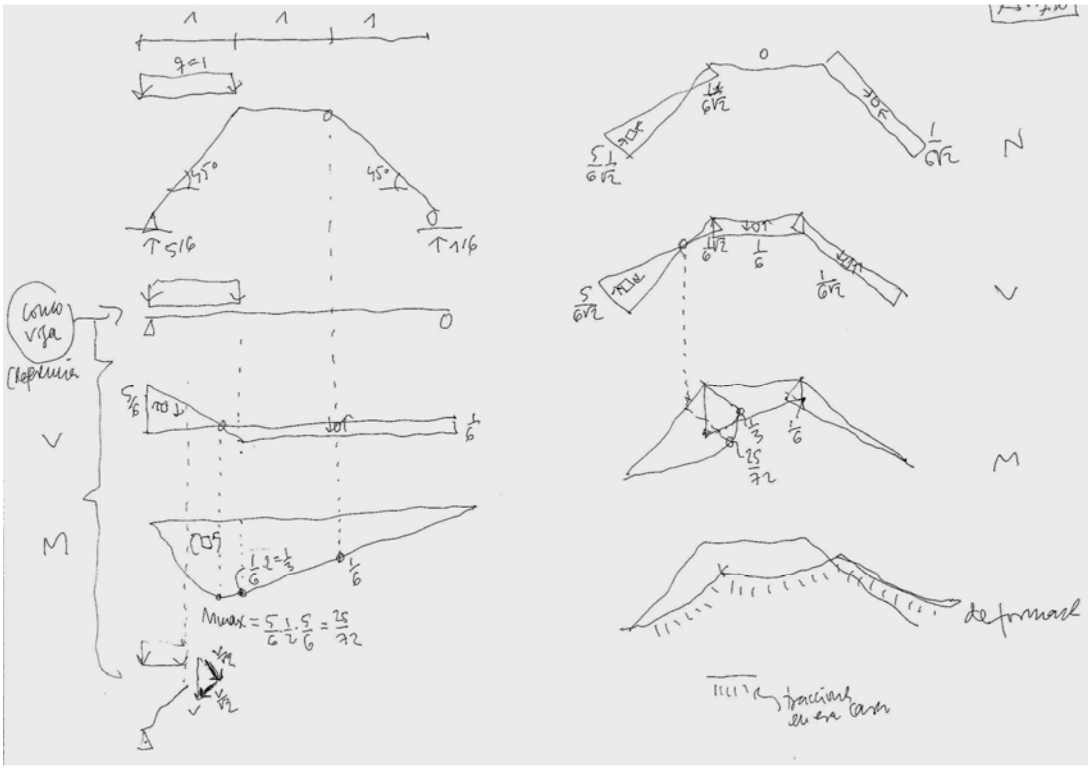
ETS de ARQUITECTURA de MADRID,
UNIVERSIDAD POLITÉCNICA DE MADRID

ESTRUCTURAS I: EJERCICIOS DE DIAGRAMAS DE ESFUERZOS



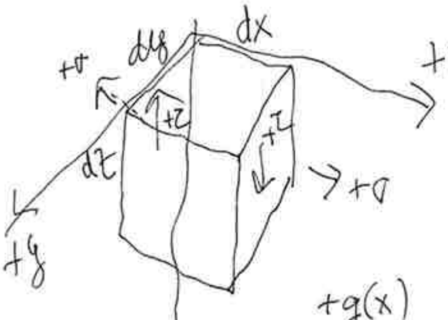
JOSÉ L. FERNÁNDEZ CABO

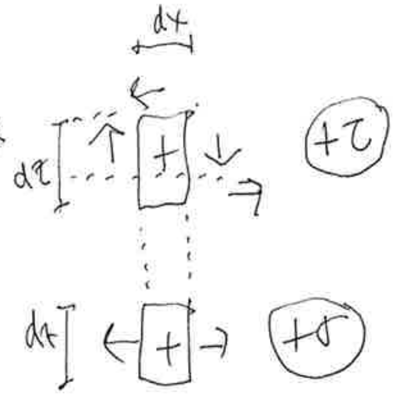
MADRID, Abril 2012 (v3)

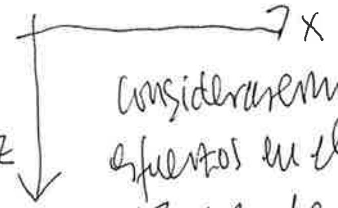
DIAGRAMAS de ESFUERZOS

Normales o Axiles (N), Cortantes (V), Momentos Flectores (M)

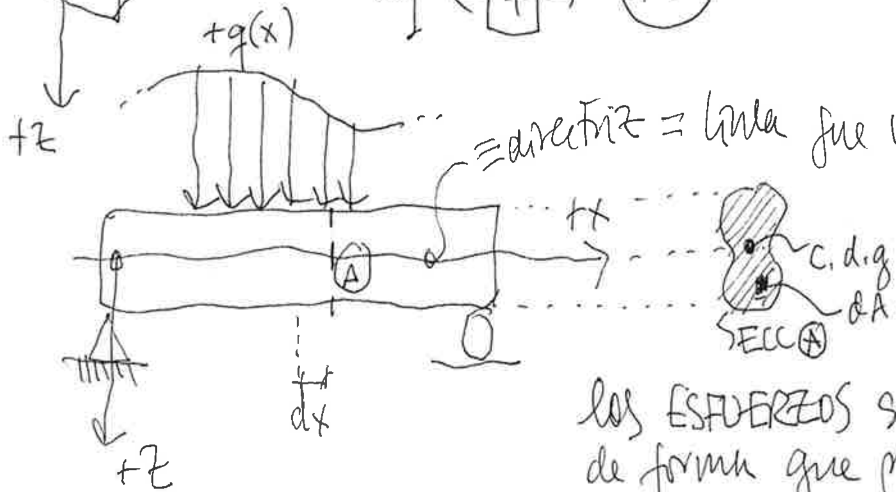
Convenio de signos $\longleftrightarrow$ vinculado al de las tensiones $\begin{cases} \text{Normales} = \sigma \\ \text{Tangenciales} = \tau \end{cases}$



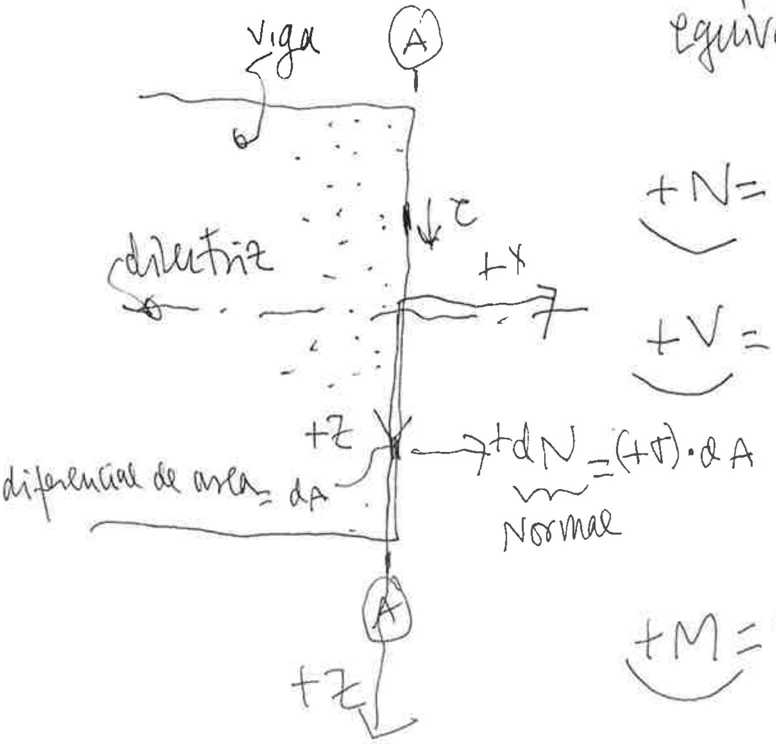




consideraremos sólo esfuerzos en el plano XZ, es decir, nos limitamos a 2D



los ESFUERZOS son INTEGRALES de TENSIONES de forma que producen un sistema equivalente de cara al EQUILIBRIO general.

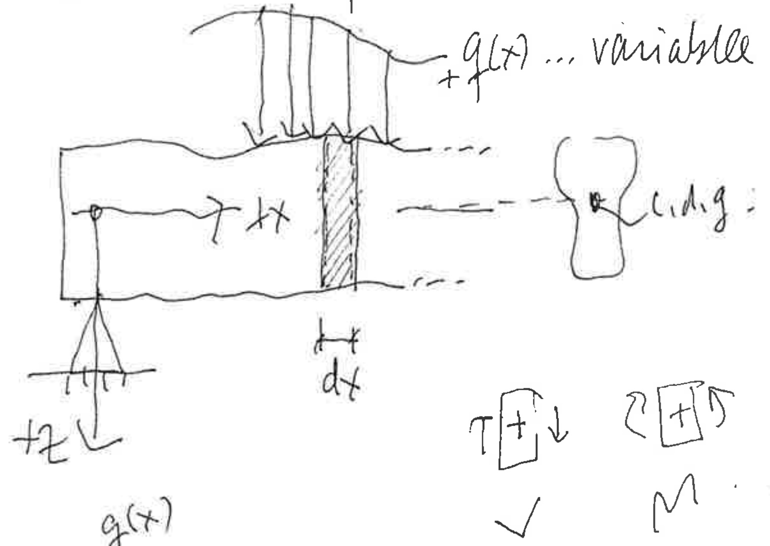


$$+N = \int +\sigma \cdot \underbrace{dA}_{\text{área}} \Rightarrow \leftarrow \boxed{+} \rightarrow \; +N$$

$$+V = \int +\tau \cdot dA \Rightarrow \uparrow \boxed{+} \downarrow \; +V$$

$$+M = \int \underbrace{\underbrace{(+\sigma)\cdot dA}_{+dN} \cdot (+z)}_{+dM} \Rightarrow \boxed{+} \; +M$$

Relaciones diferenciales entre cortante y flector (→FLEXION)

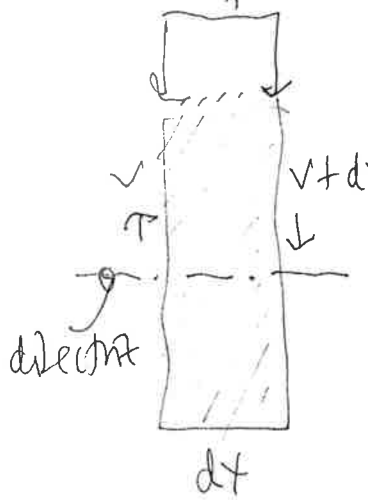


+x

+Z

+↓ q(x) = { Acciones... carga lineal...

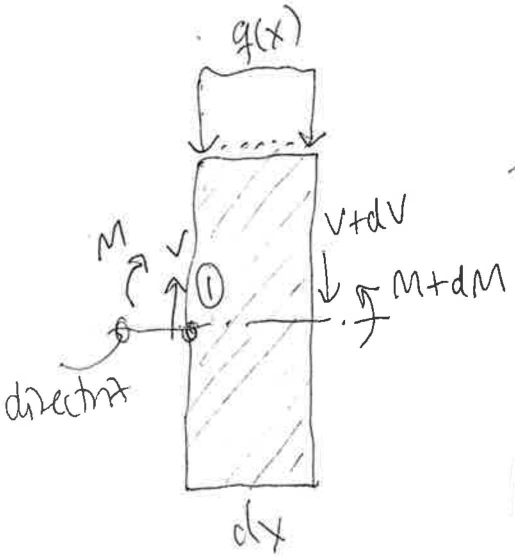
+↓ w(x) = descenso



$\sum F_Z = 0 \quad +\downarrow$

$q\cdot dx \cancel{- V} + \cancel{V} + dV = 0 \Rightarrow q\,dx + dV = 0$

$\boxed{q = \frac{-dV}{dx}}$ ... ó ... $\underbrace{V_{AB} = \int_A^B -q\cdot dx}_{\text{no suele usarse}}$



$\sum M_{1} = 0 \quad +\circlearrowright$

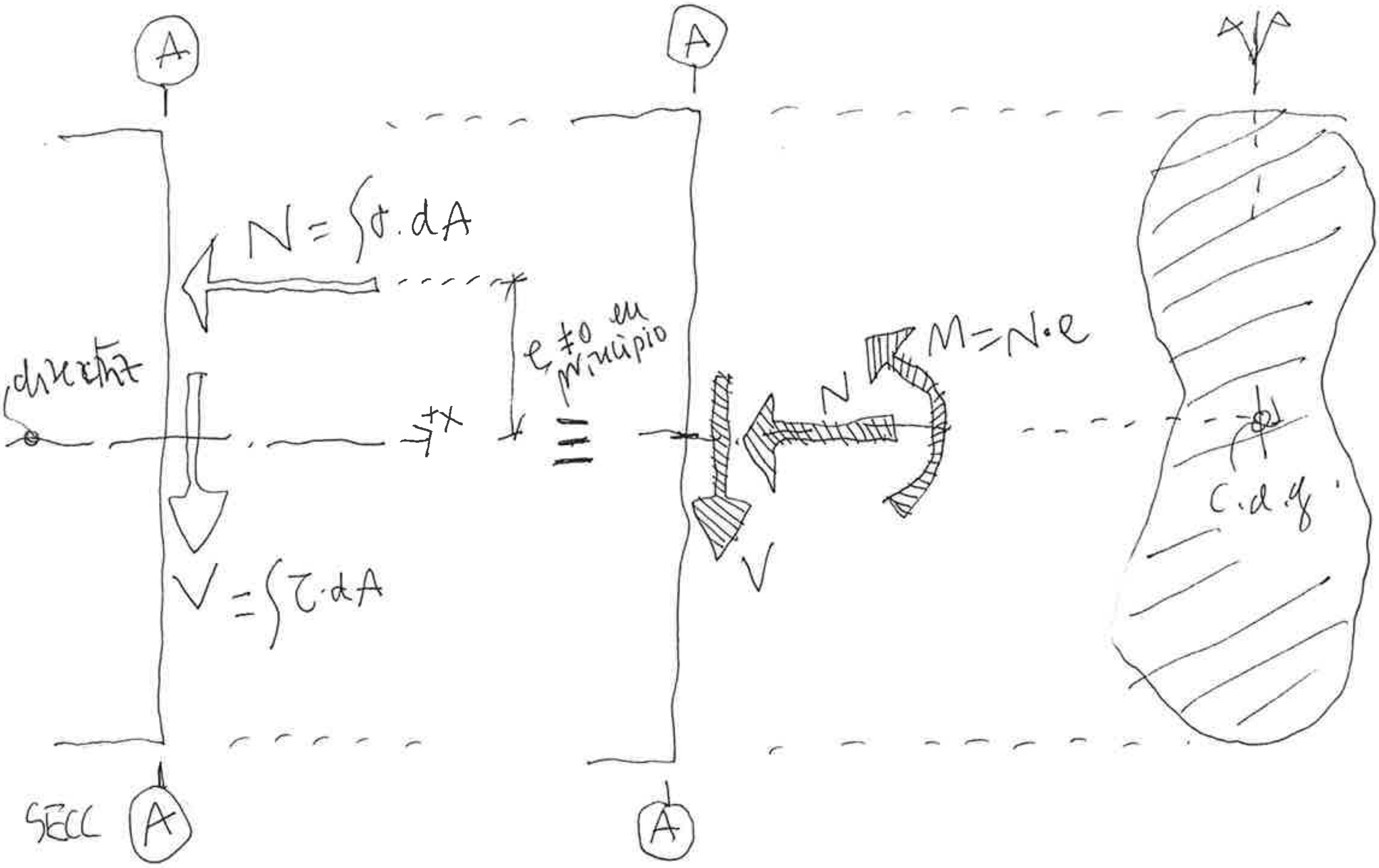
$\cancel{M} - \cancel{M} - dM + \overbrace{\cancel{q\cdot dx\cdot\frac{dx}{2}}}^{2^{\circ}\text{ orden}} + V\cdot dx + \overbrace{\cancel{dV\cdot dx}}^{2^{\circ}\text{ orden}} = 0$

$-dM + V dx = 0$

$\boxed{\frac{dM}{dx} = V}$ ... o $\boxed{M_{AB} = \int_A^B V\cdot dx}$

diferencia o salto de momento entre dos secciones A y B.

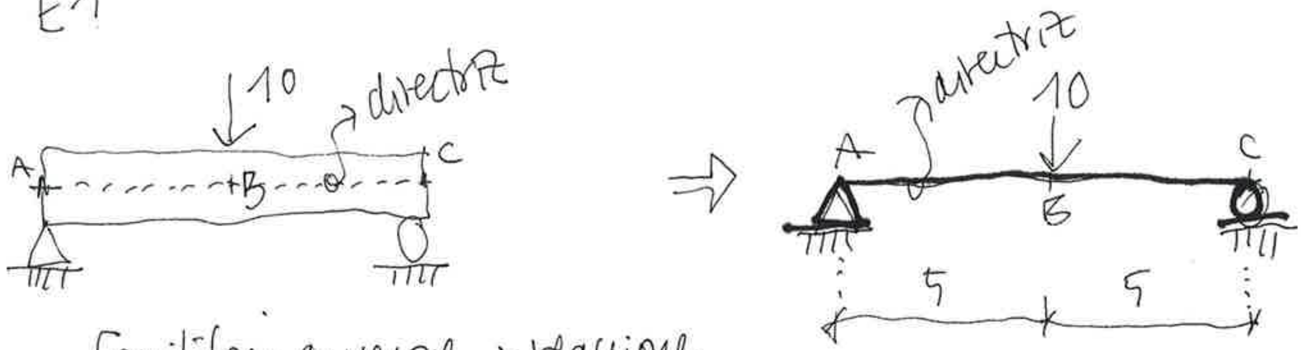
el normal <u>N no interviene</u> ni en $\sum F_Z = 0$ ni en $\sum M = 0$, ya que en el segundo caso además es con referencia a la directriz



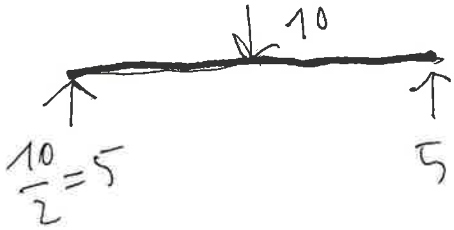
SOLO son necesarios (y suficientes) tres esfuerzos, resultantes de las tensiones σ y τ, tomando además como referencia la directriz de la pieza.

<u>N, M y V representan</u>, en forma de tres resultantes equivalentes, las fuerzas que, en ese corte, y de cara al equilibrio general (no a nivel diferencial) ejerce la parte de la estructura que es eliminada sobre dicho corte.

E1



Equilibrio general → reacciones



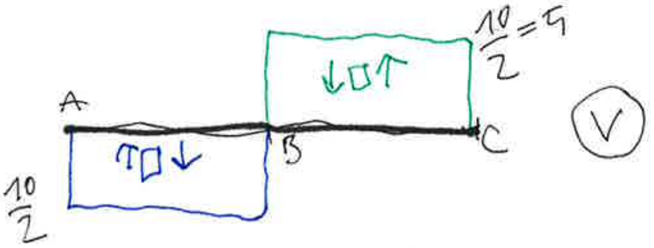
... simetría ... efectivamente:

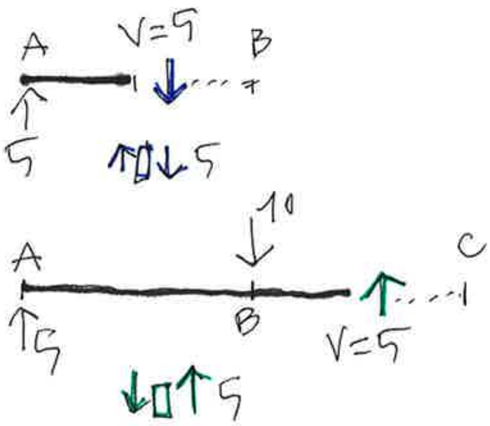
$\sum M_{(A)} = 0$

$10 \cdot 5 - 5 \cdot 10 = 0$

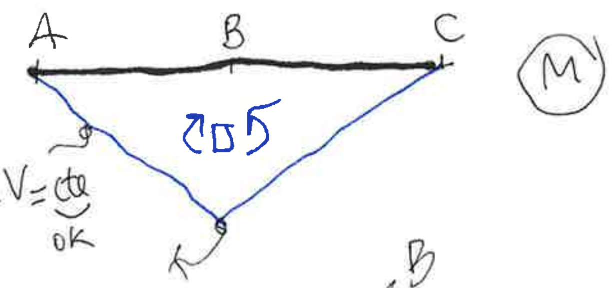
$\sum F_z = 0 \Rightarrow 10 - 5 - 5 = 0$

Esfuerzos.

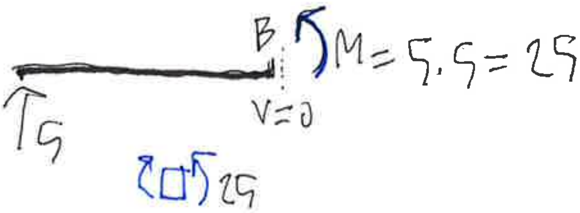




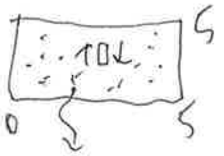
$\frac{dV}{dx} = -q = 0$ ok



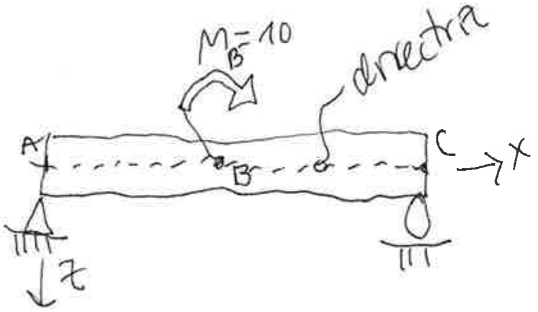
$\frac{dM}{dx} = V = cte$ ok

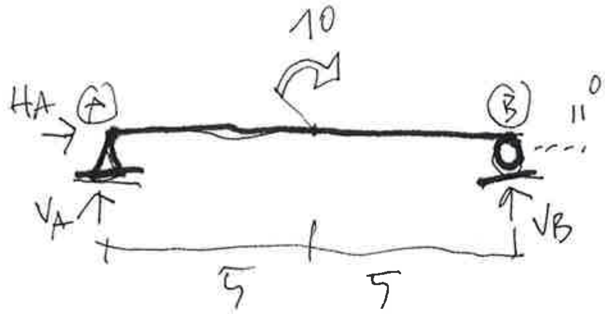


$$25 = M_{AB} = \int_A^B V\, dx$$

$M_{AB} =$ 

área gráfica de cortantes entre las secciones A y B





Equilibrio general → reacciones.

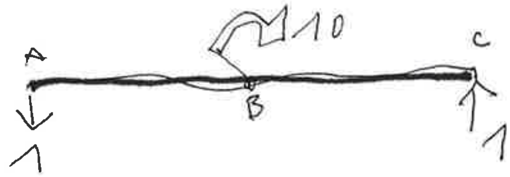
$\sum F_x = 0 \Rightarrow H_A = 0$

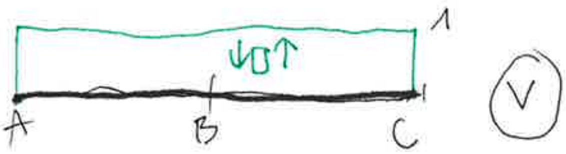
$\sum M_{A} = 0 \;\; 10 - V_B \cdot 10 = 0$

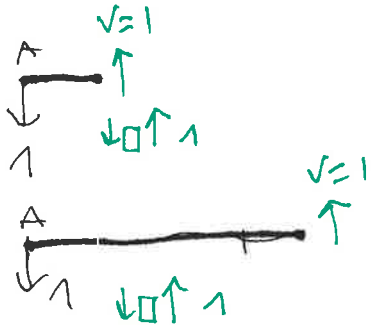
$V_B = 1$

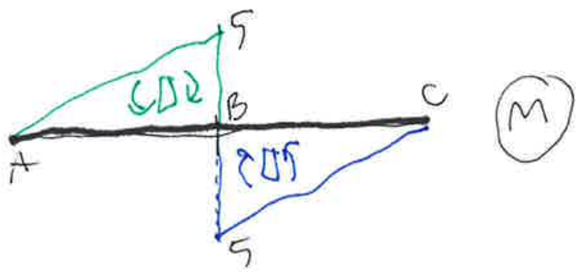
$\sum F_z = 0 \Rightarrow V_A + V_B = 0$

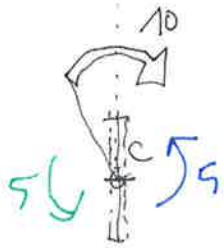
$V_A = -V_B = -1$



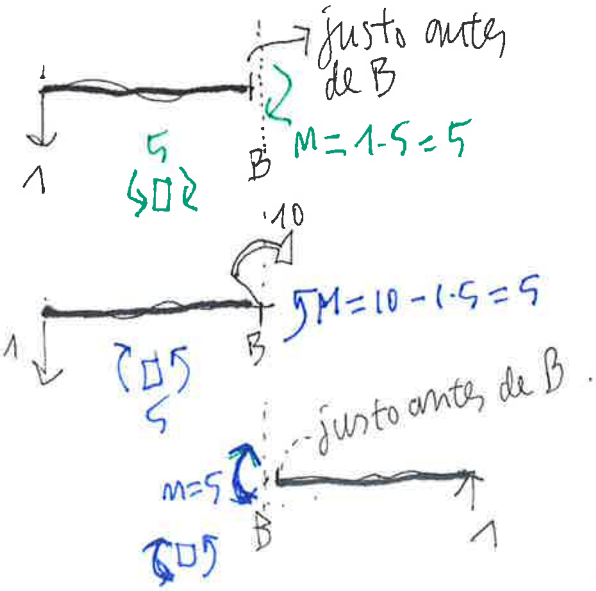




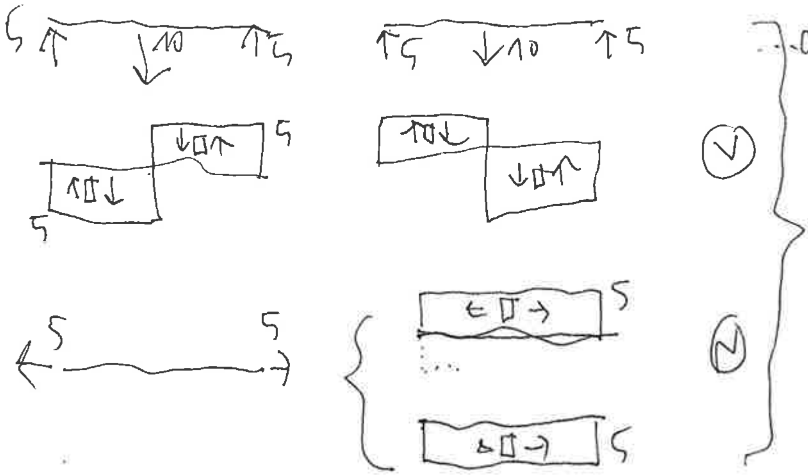




comprobación del equilibrio en C

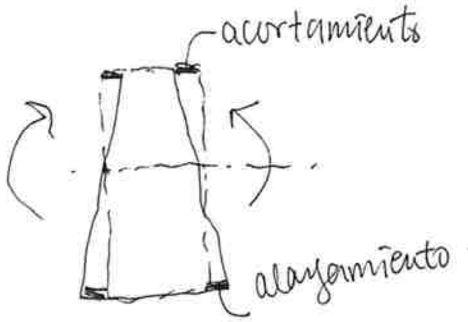


convenios para dibujar las gráficas



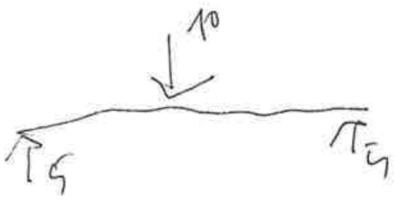
ambos validos a priori
(salvo indicación en contra expresa)

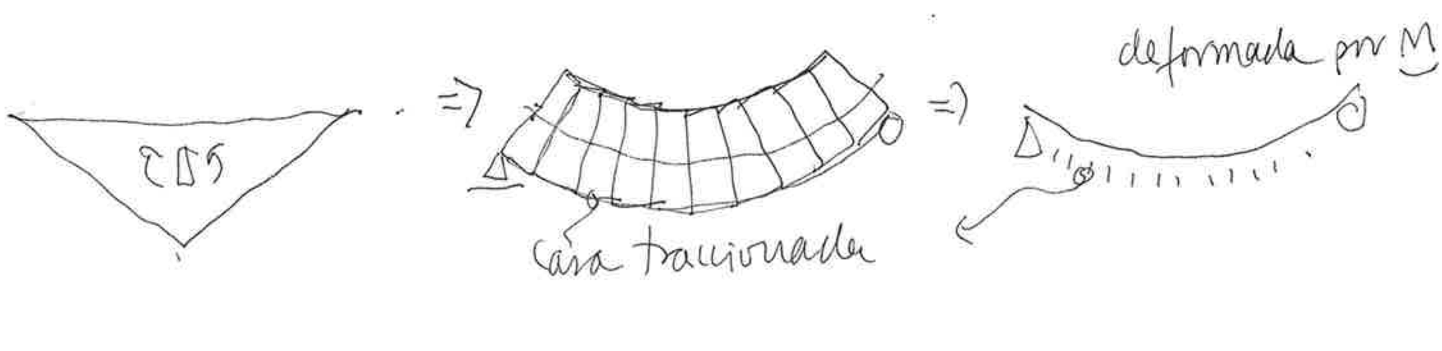
PERO con los FLECTORES SÍ HAY CONVENIO general.



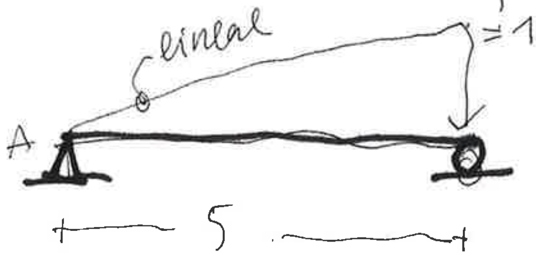
la gráfica se dibuja siempre
hacia el lado en donde estarían
los alargamientos, la cara traccionada.

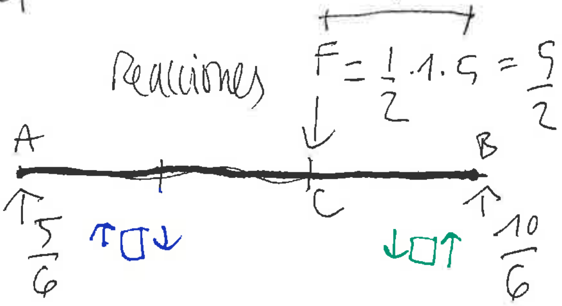
es decir.



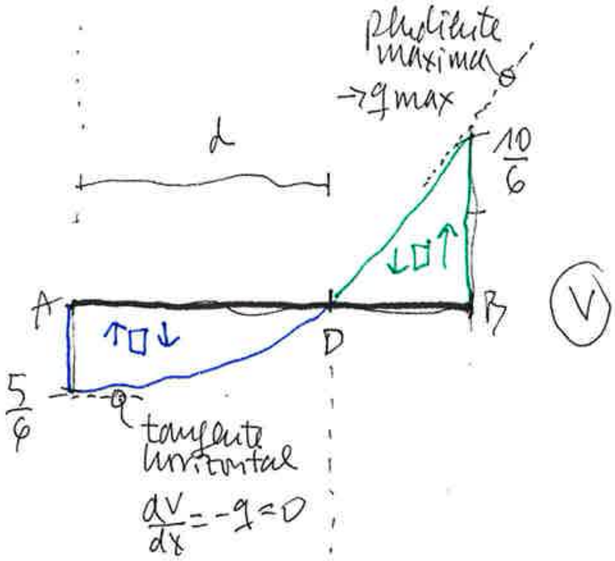


Uso de relaciones diferenciales.

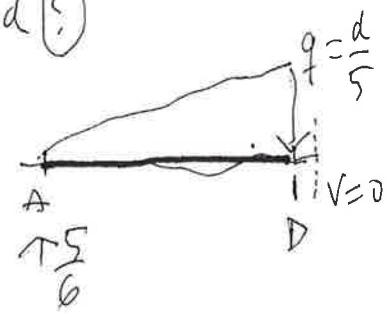




q = lineal $\Rightarrow$ V = cuadrática $\Rightarrow$ M $\Rightarrow$ cúbica.



d ?

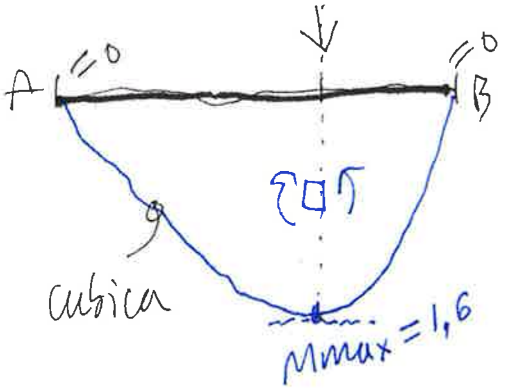


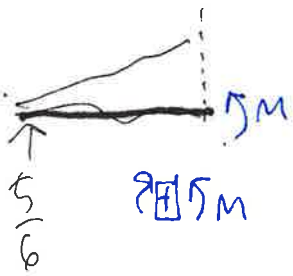
$\sum F_z = 0$

$$\frac{5}{6} = \frac{1}{2} d \cdot \frac{d}{5}$$

$$d = \sqrt{50/6} = 2{,}89$$

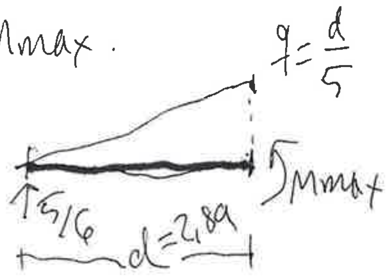
si $M_{max} \Rightarrow V = 0$; $\frac{dM}{dx} = V$





cálculo de M_{max}.

- por corte.

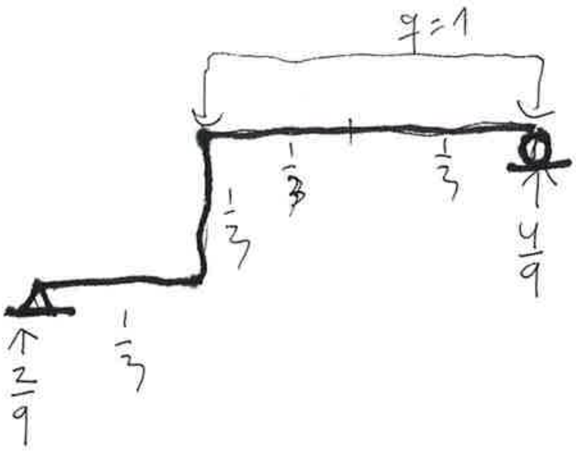


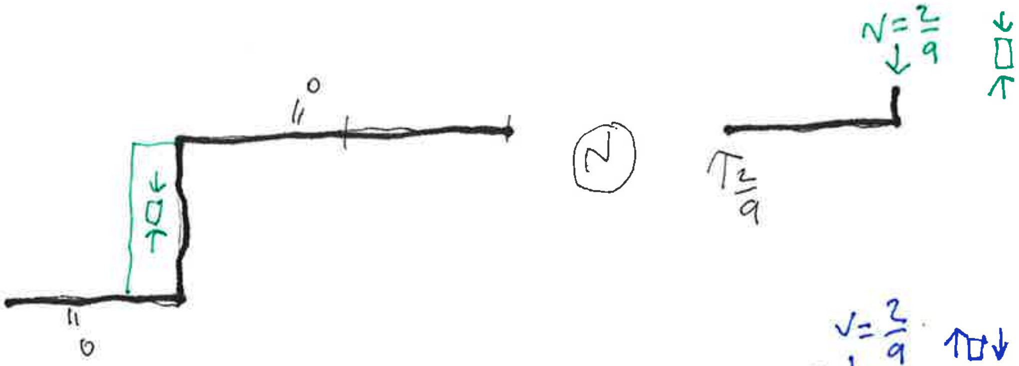
$$M_{max} = \frac{5}{6} \cdot 2{,}89 - \frac{1}{2}\, 2{,}89 \cdot \frac{2{,}89}{5} \cdot \frac{2{,}89}{3} = 1{,}6$$

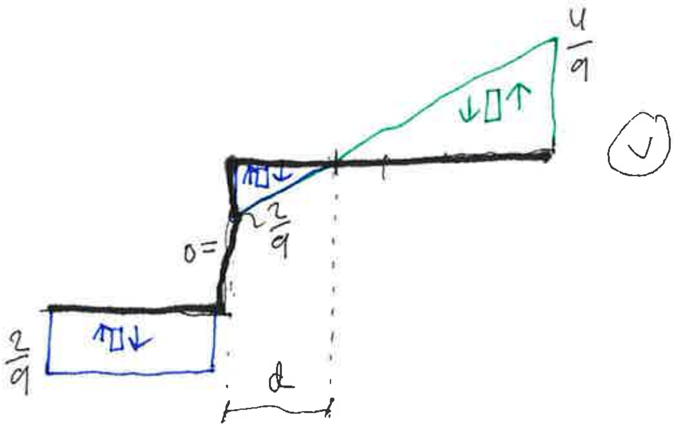
- integrando área de cortantes.

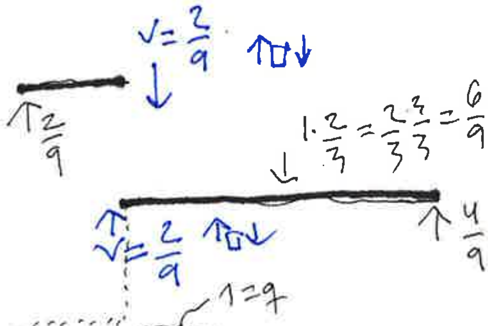
$M_{max} =$

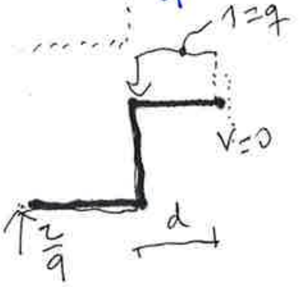
$= \frac{2}{3} \cdot \frac{5}{6} \cdot 2{,}89 = 1{,}6$



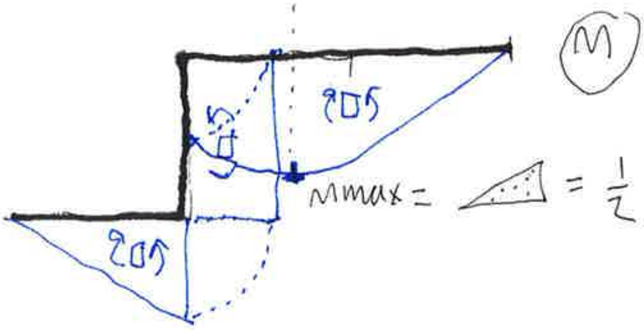




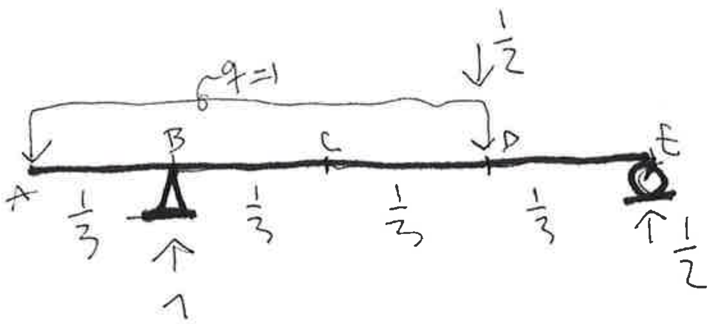


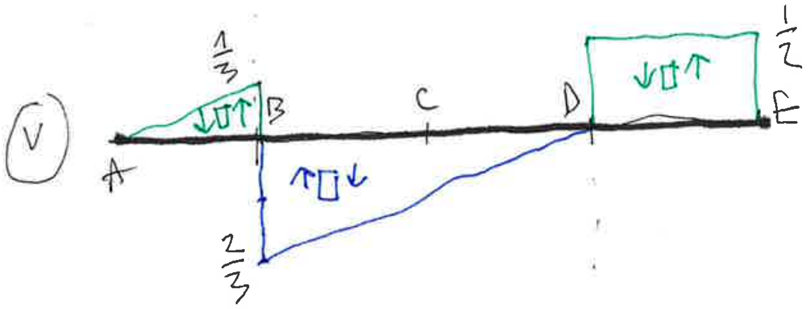


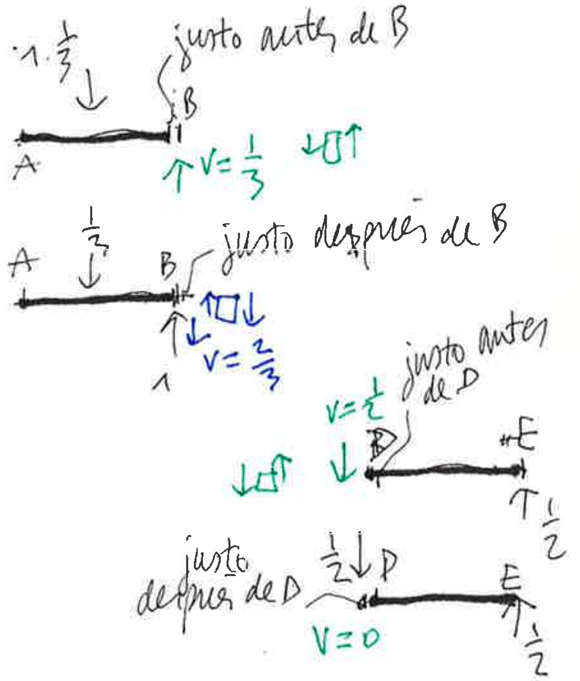
$$q \cdot d = \frac{2}{9} \Rightarrow d = \underbrace{\frac{2}{9}}_{} = \frac{1}{4{,}5}$$

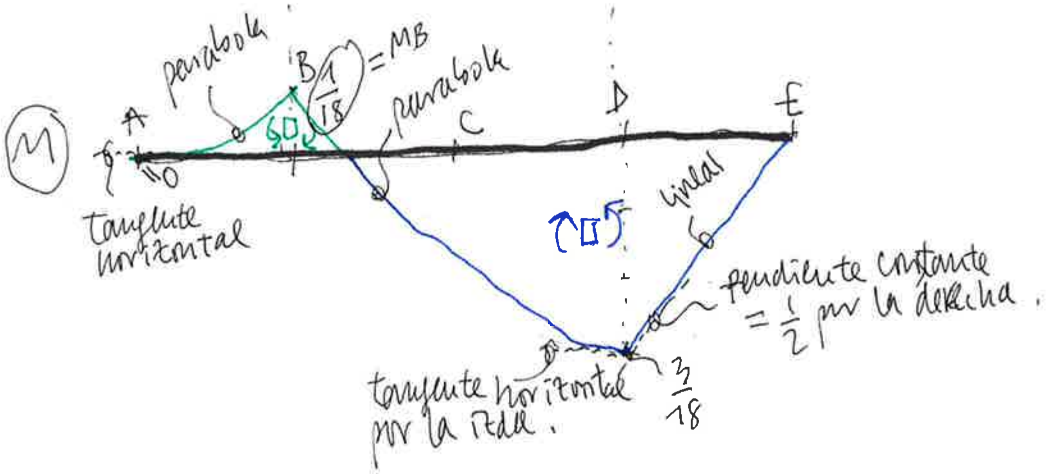


deformada
por el flector.









Integrando areas de cortantes.

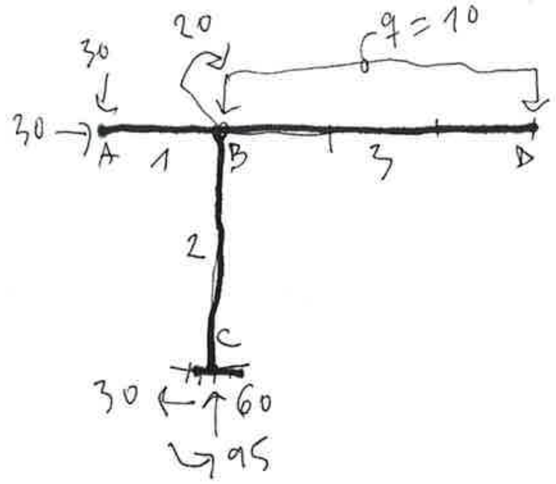
$$M_B = \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{3} = \frac{-1}{18}$$

$$M_D = \frac{-1}{18} + \ldots = \frac{-1}{18} + \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{2}{3} = +\frac{3}{18}$$

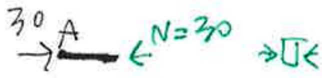
Si ahora se computa el area de cortantes por la derecha

$$M_D \equiv \int_E^D V\cdot dx = \ldots = \frac{1}{2}\cdot\frac{1}{3} = \frac{3}{3}\cdot\frac{1}{2}\cdot\frac{1}{3} = \frac{3}{18}$$

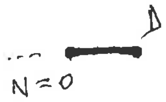
ok



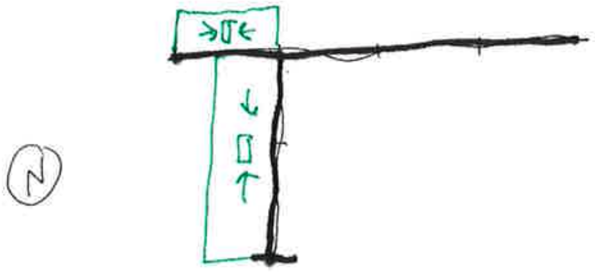
20
q = 10
30
30
A
1
B
3
D
2
C
30
60
95



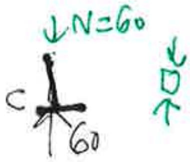
30
A
N = 30



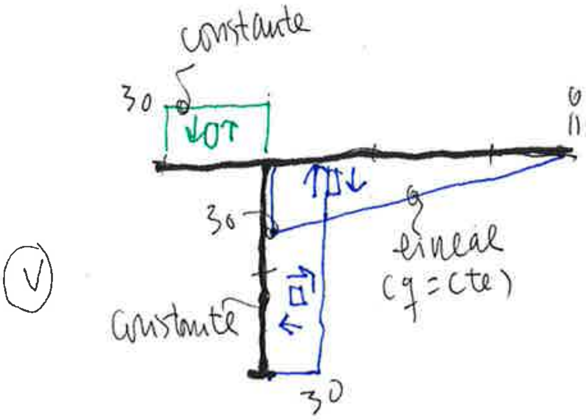
D
N = 0



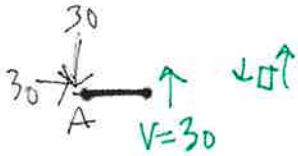
N



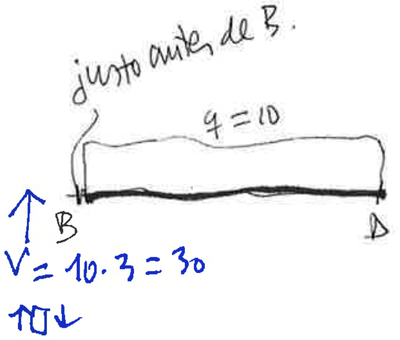
N = 60
C
60



constante
30
0
30
lineal
(q = cte)
constante
30
V

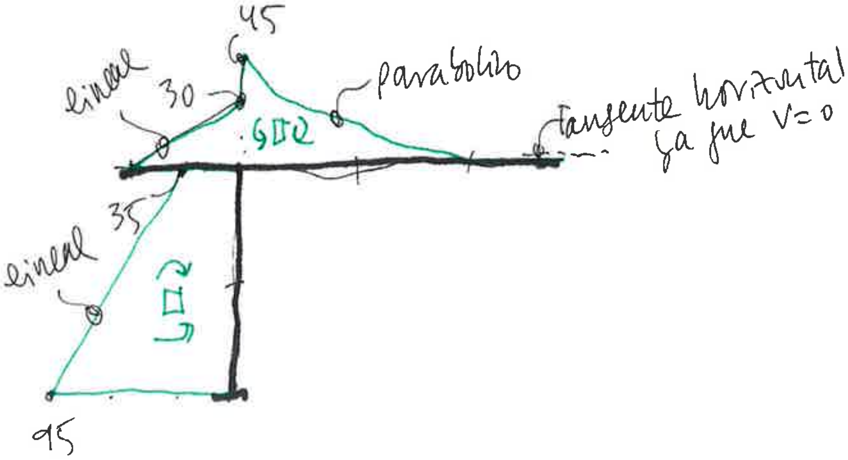


30
30
A
V = 30

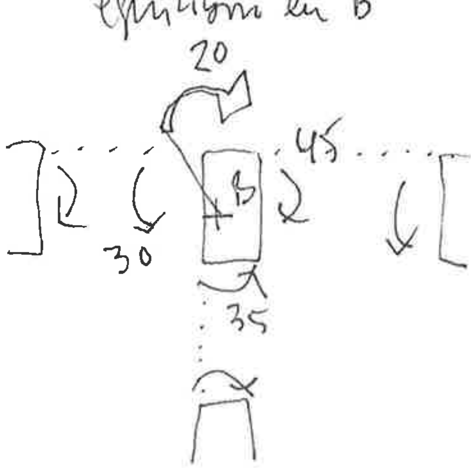


justo antes de B.
q = 10
B
D
V = 10 · 3 = 30

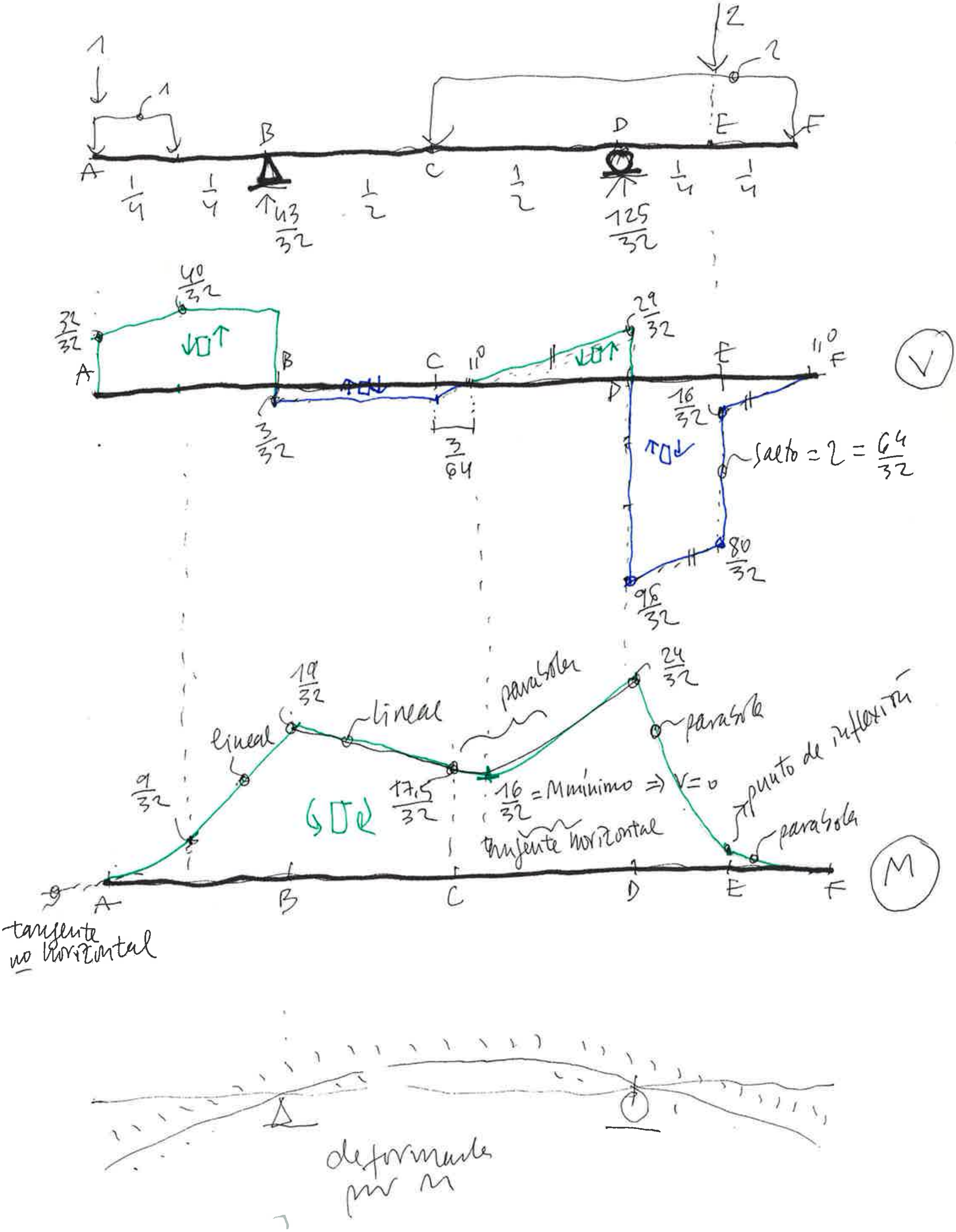
V = 30
30
C
60
95

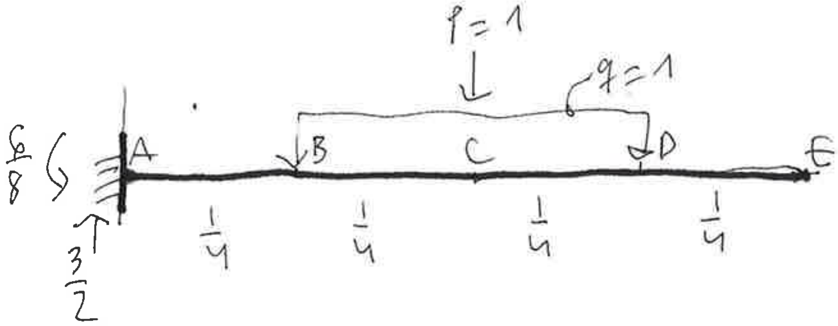


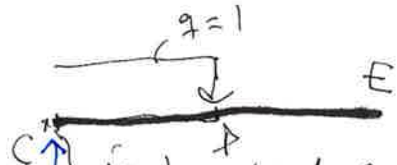
45
lineal
30
parabólica
tangente horizontal
ya que V = 0
35
lineal
95



equilibrio en B
20
45
B
30
35

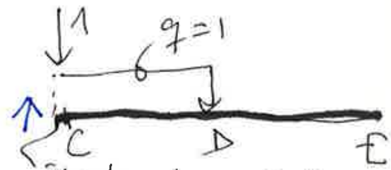

1
1
12
2
A
B
C
D
E
F
1/4
1/4
1/2
1/2
1/4
1/4
43/32
125/32
40/32
32/32
29/32
11°
3/32
3/64
16/32
salto = 2 = 64/32
80/32
96/32
V
19/32
24/32
lineal
lineal
parabola
parabola
parabola
9/32
17,5/32
16/32 = Mmínimo ⇒ V = 0
tangente horizontal
punto de inflexión
tangente no horizontal
M
deformada por M





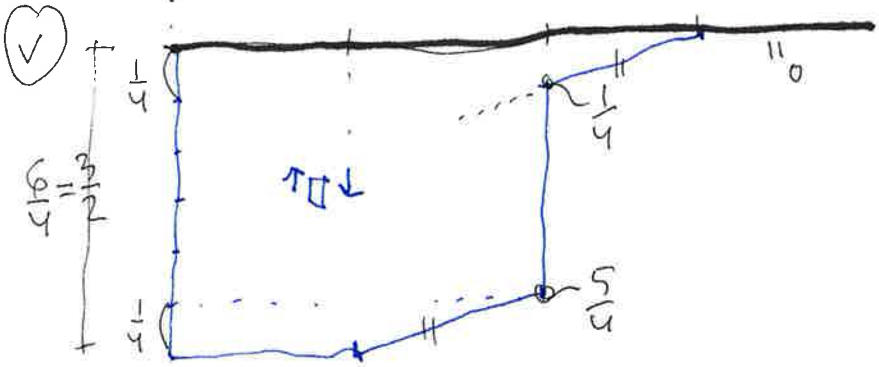
justo antes de C

$V = 1 - \frac{1}{4} = \frac{1}{4}$ ↑▯↓



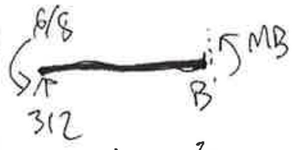
$V = 1 + \frac{1}{4} = \frac{5}{4}$ ↑▯↓

justo después de C

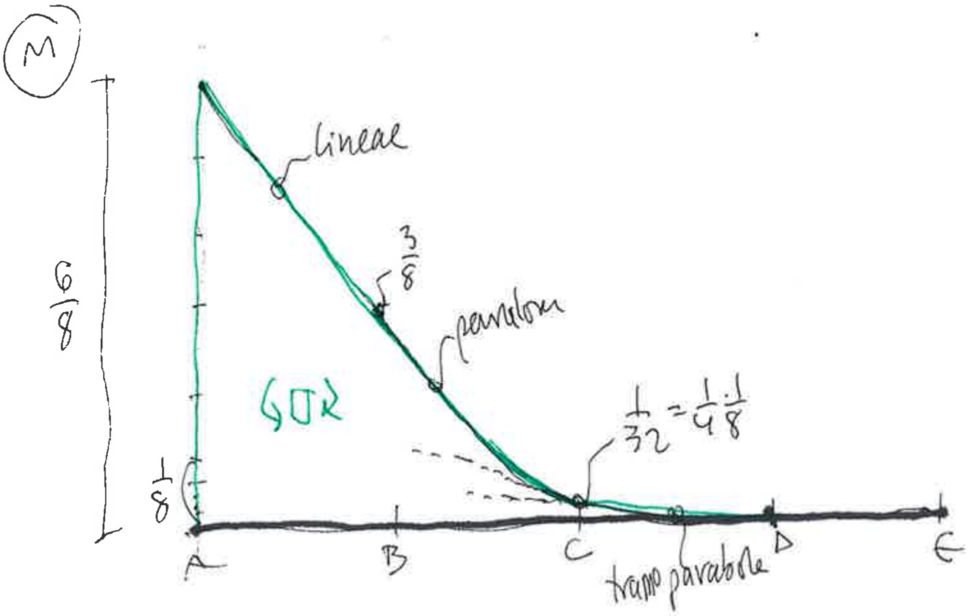


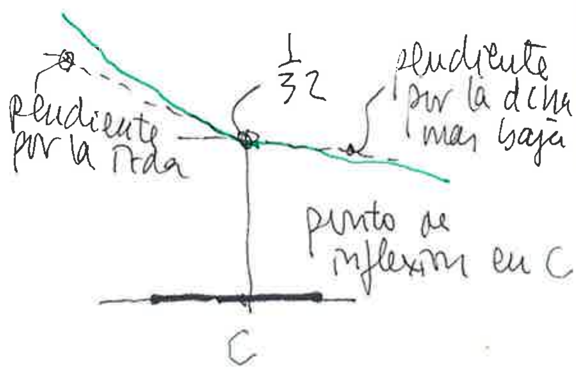
$M_B = \frac{6}{8} - \boxed{\frac{1}{4} \cdot \frac{6}{4}} = \frac{6}{8} - \frac{1}{4} \cdot \frac{6}{4} = \frac{12-6}{16} = \frac{3}{8}$

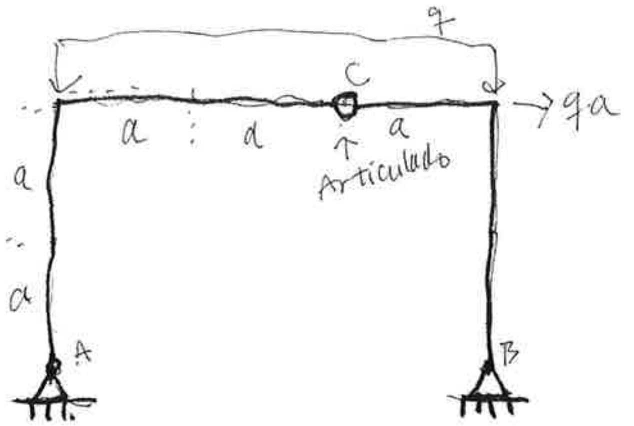
o también

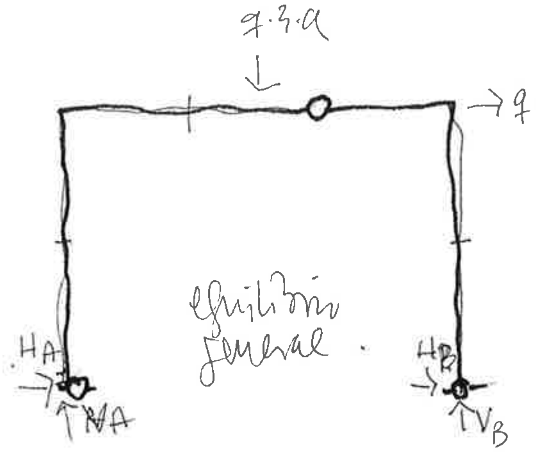


$M_B = \frac{6}{8} - \frac{3}{2} \cdot \frac{1}{4} = \frac{3}{8}$







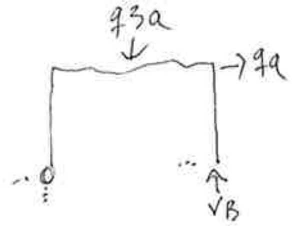


sentido arbitrario a priori

$\Sigma M_{A} = 0$ +↻

$q \cdot 3 \cdot a \cdot \frac{3a}{2} + q \cdot a \cdot 2a - V_B \cdot 3a = 0$

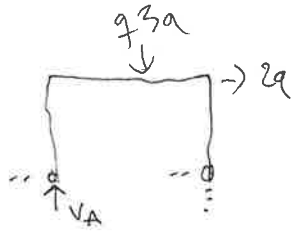
$qa^2 \left(\frac{9}{2} + 2 \cdot \frac{2}{2}\right) = 3a V_B \Rightarrow \underline{V_B = \frac{13}{6} q \cdot a}$



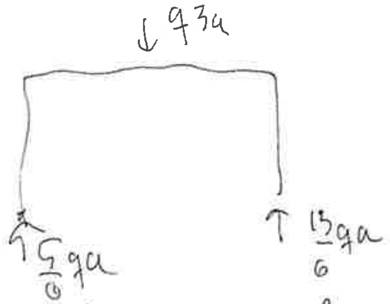
$\Sigma M_{B} = 0$ ↺+

$q \cdot 3a \cdot \frac{3a}{2} - qa \cdot 2a - V_A \cdot 3a = 0$

$qa^2 \left(\frac{9}{2} - 2 \cdot \frac{2}{2}\right) = 3a V_A \Rightarrow \underline{V_A = \frac{5}{6} qa}$.

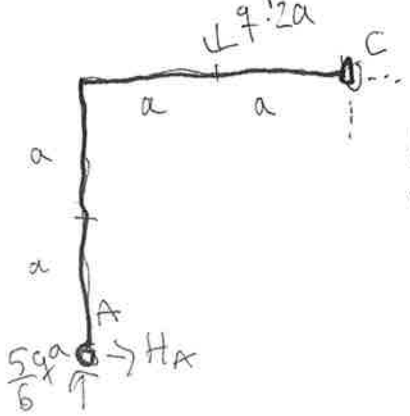


¿equilibrio vertical y horizontal de fuerzas.



$\left(\frac{5}{6} + \frac{13}{6}\right) qa = \frac{18}{6} qa = \underset{ok}{3qa}$ $\quad \Sigma F_z = 0$ ok

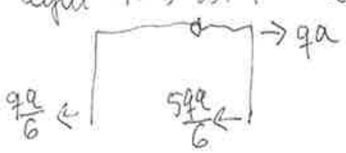
aislando partes salen H_A y H_B

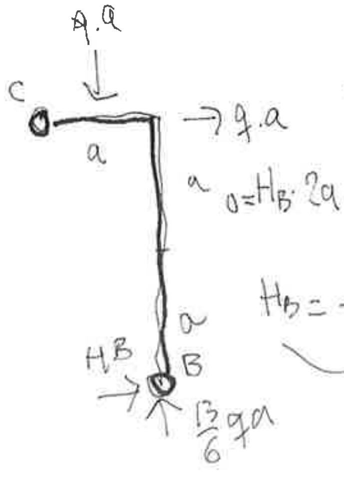


$\Sigma M_{C} = 0$ +↻

$\underline{H_A = \frac{5}{6} qa - qa = -\frac{qa}{6}}$

equilibrio horizontal ¡ok



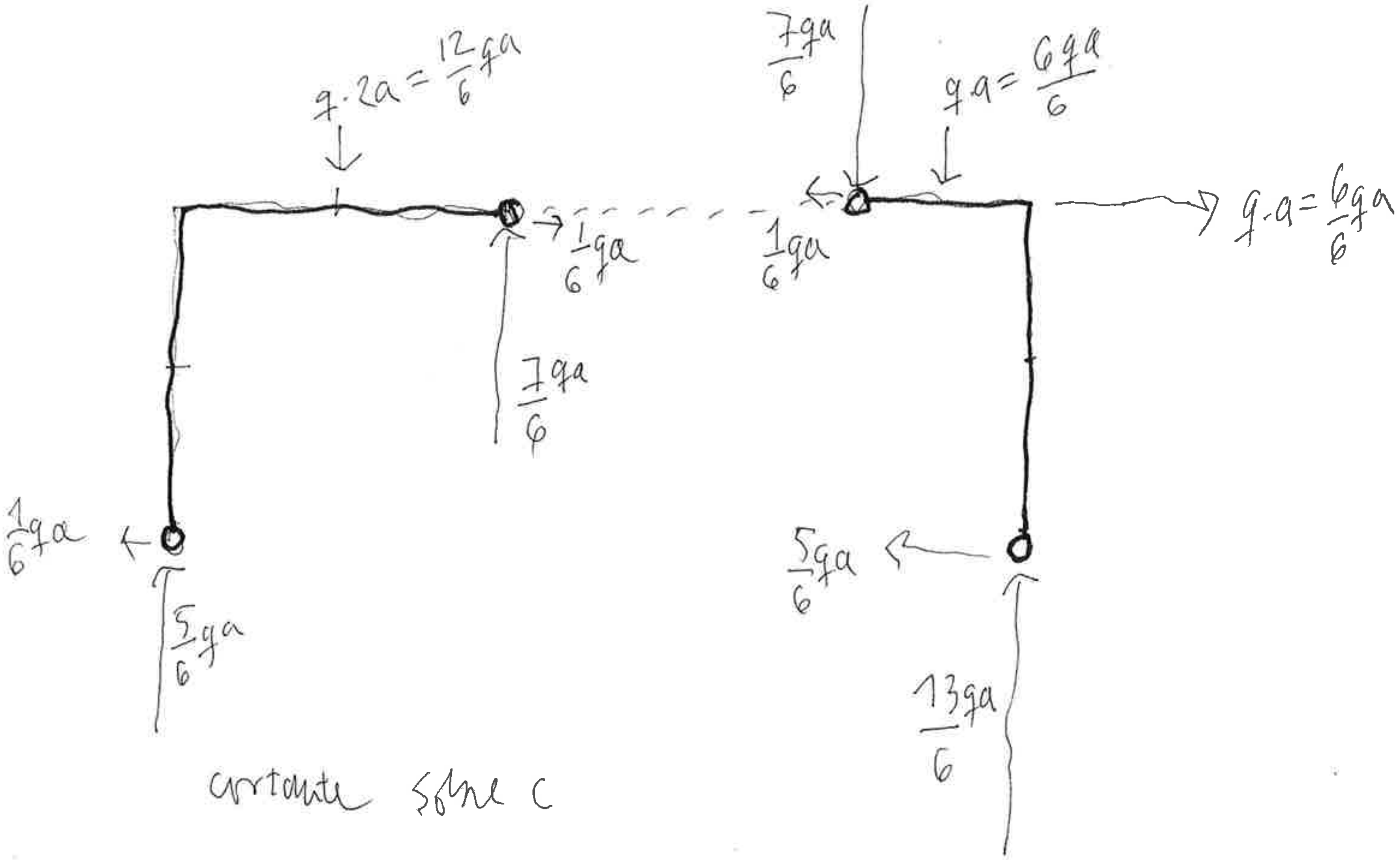


$\Sigma M_{C} = 0$ ↻

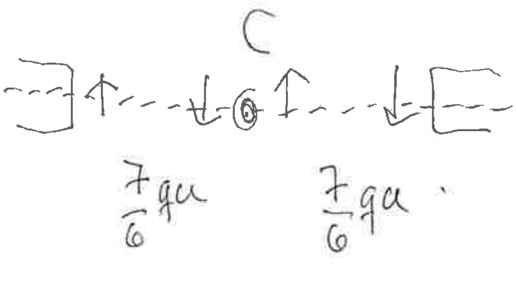
$0 = H_B \cdot 2a + \frac{13}{6} qa^2 - \frac{qa^2}{2}$

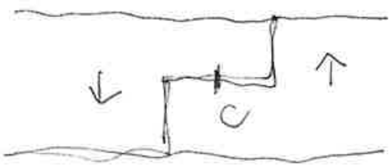
$\underline{H_B = -\frac{5}{6} qa}$

esfuerzo en la articulación centro C



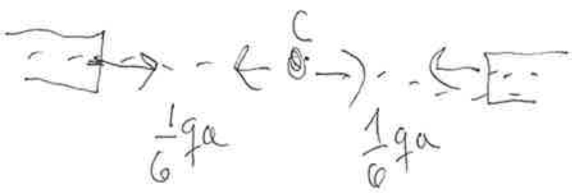
cortante sobre C

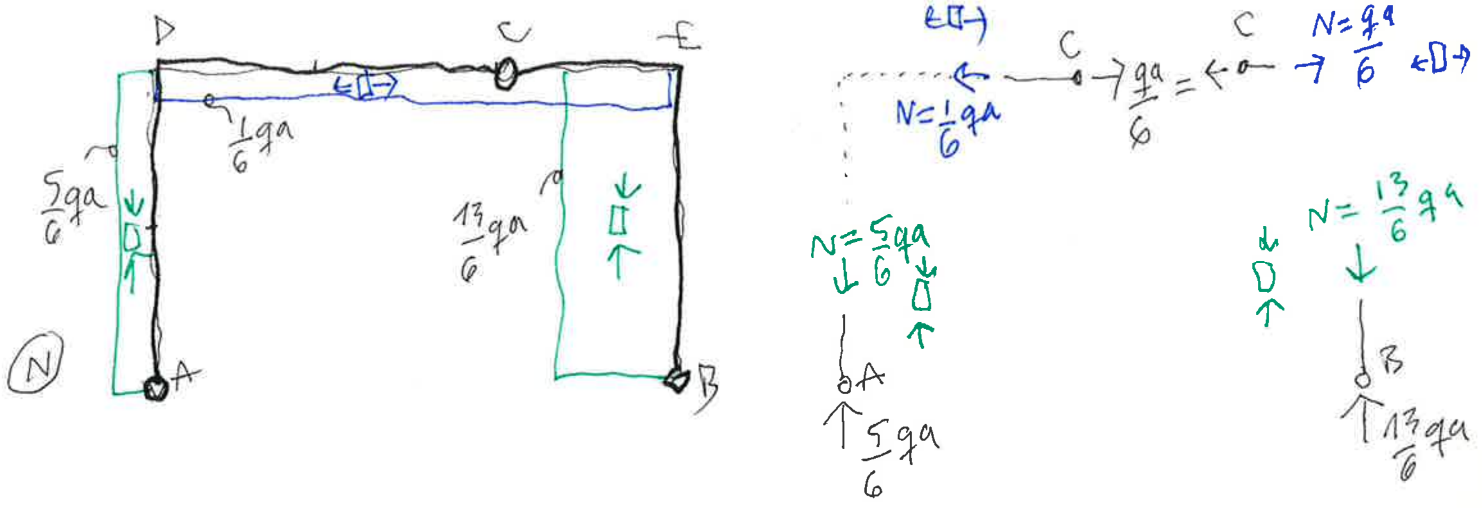


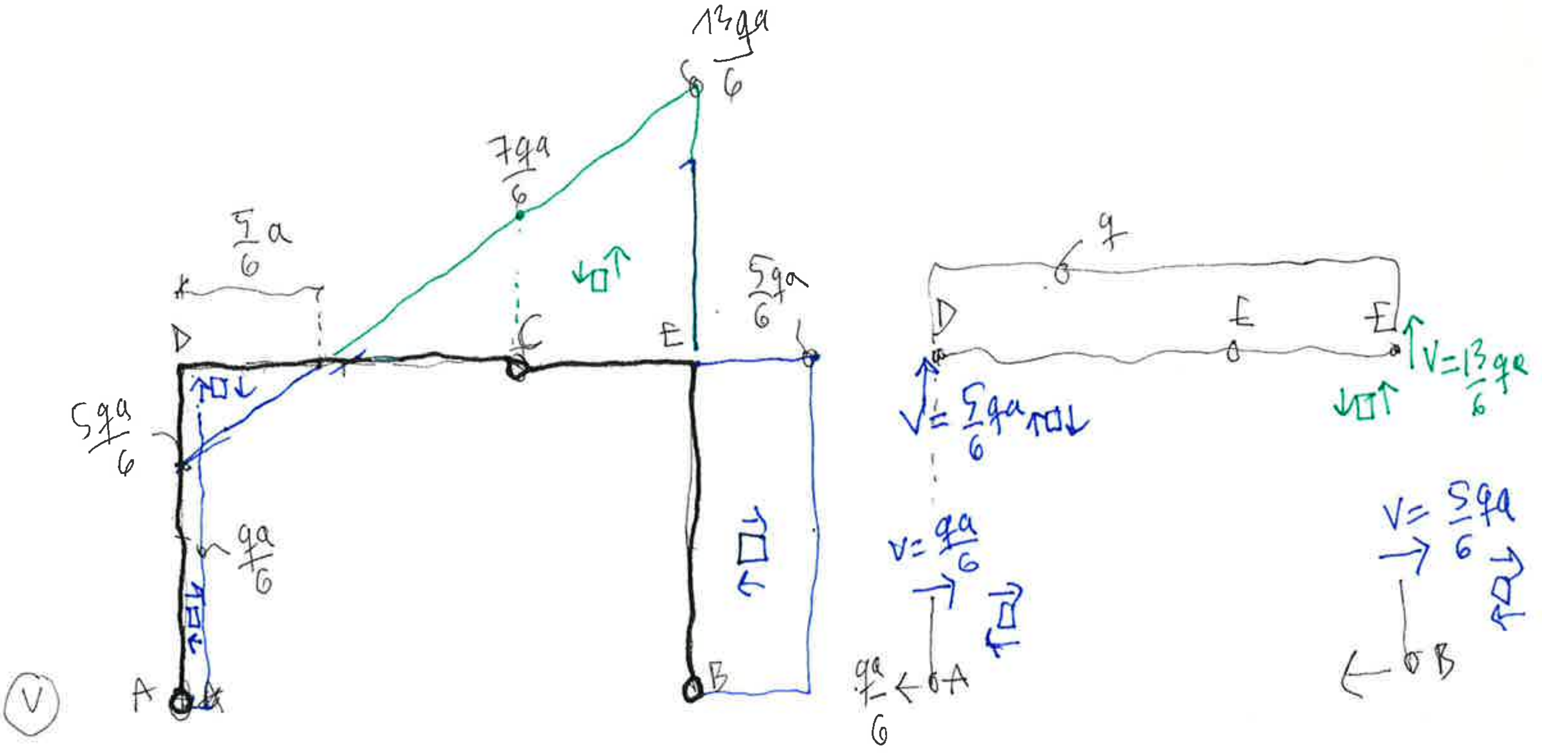


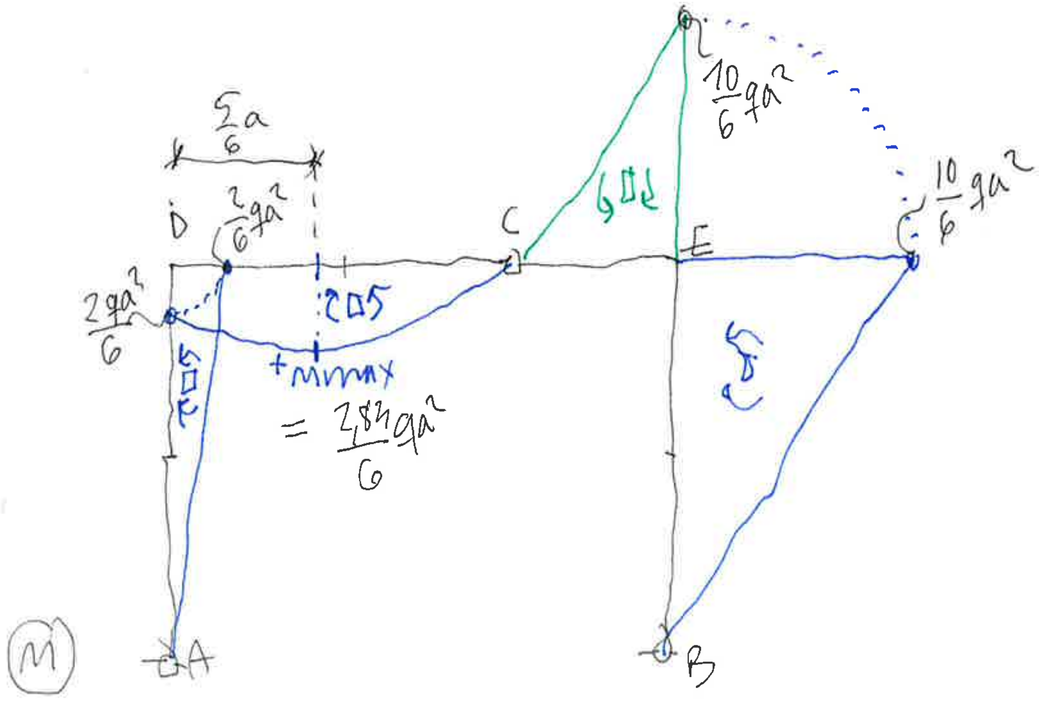
valdría media madera de este modo.

pero hay tracción también sobre ella

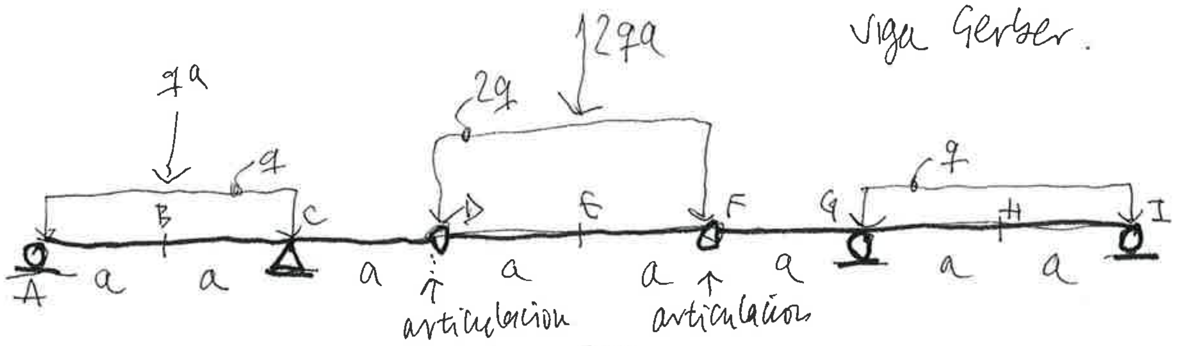



N
D
C
E
A
B
$\frac{1}{6}qa$
$\frac{5}{6}qa$
$\frac{13}{6}qa$
$N=\frac{1}{6}qa$
$\frac{7qa}{6}$
$N=\frac{qa}{6}$
$N=\frac{5}{6}qa$
$N=\frac{13}{6}qa$
$\frac{5}{6}qa$
$\frac{13}{6}qa$

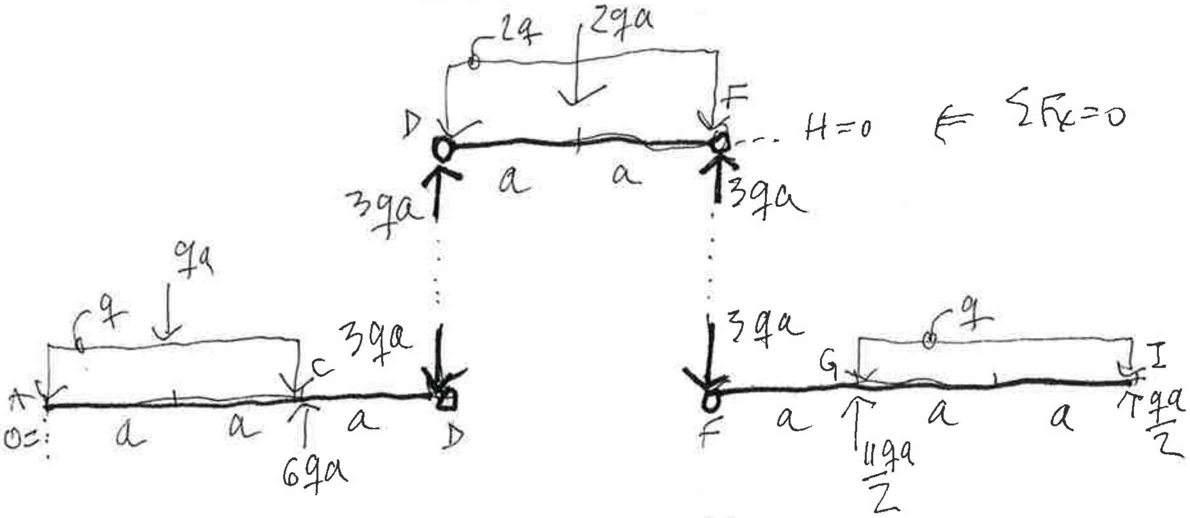

V
$\frac{13qa}{6}$
$\frac{7qa}{6}$
$\frac{5}{6}a$
$\frac{5qa}{6}$
$\frac{5qa}{6}$
$\frac{qa}{6}$
D
C
E
A
B
q
$V=\frac{13}{6}qa$
$V=\frac{5}{6}qa$
$V=\frac{qa}{6}$
$V=\frac{5qa}{6}$
$\frac{qa}{6}$


M
$\frac{10}{6}qa^2$
$\frac{10}{6}qa^2$
$\frac{5a}{6}$
$\frac{2}{6}qa^2$
$\frac{2qa^2}{6}$
D
C
E
A
B
$+M_{max} = \frac{2{,}84}{6}qa^2$

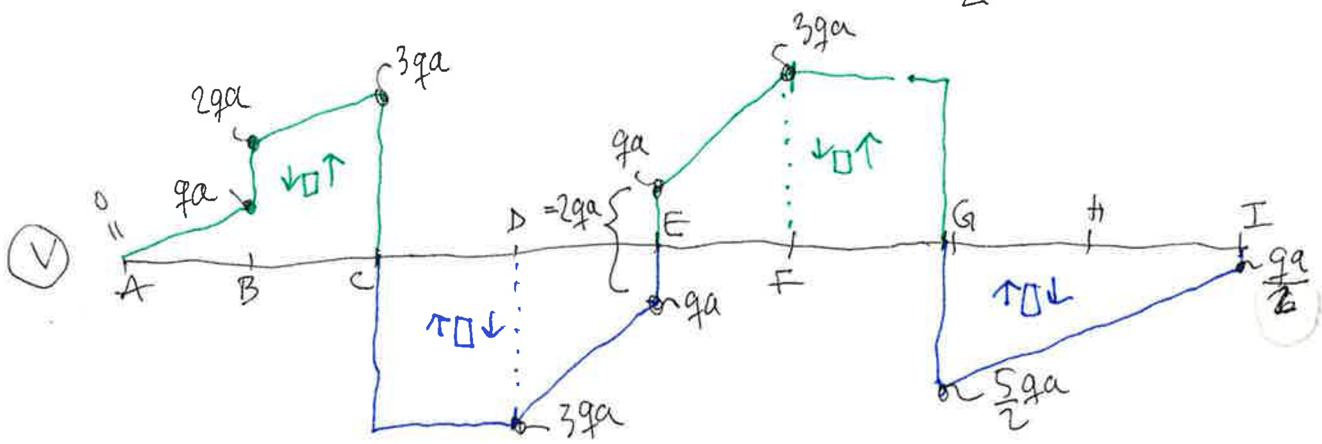
viga Gerber.



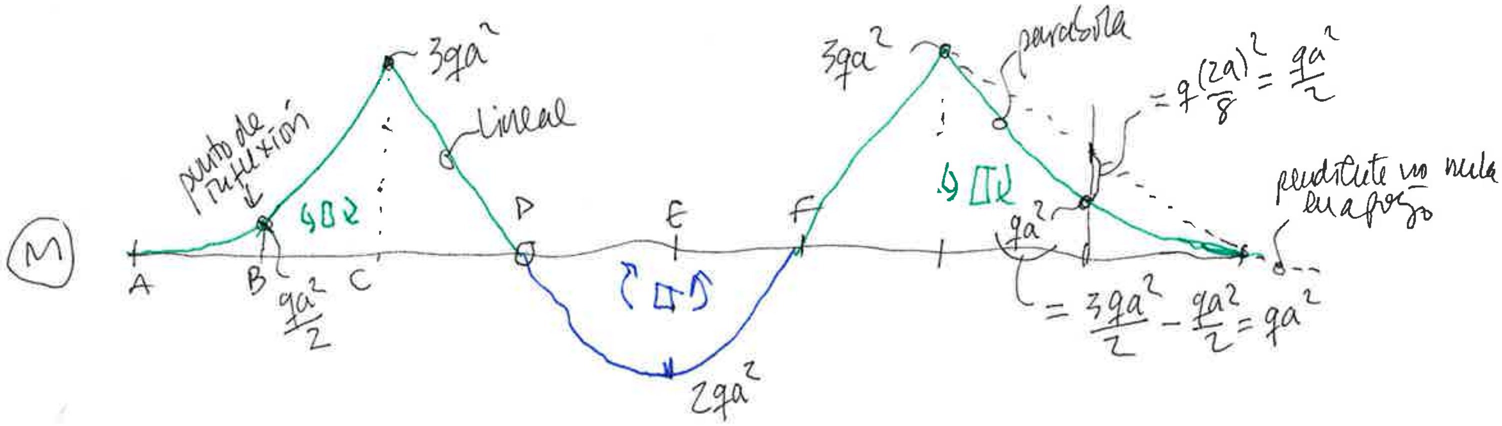
12qa
qa
2q
q
q
A B C D E F G H I
a a a a a a a a
articulación
articulación



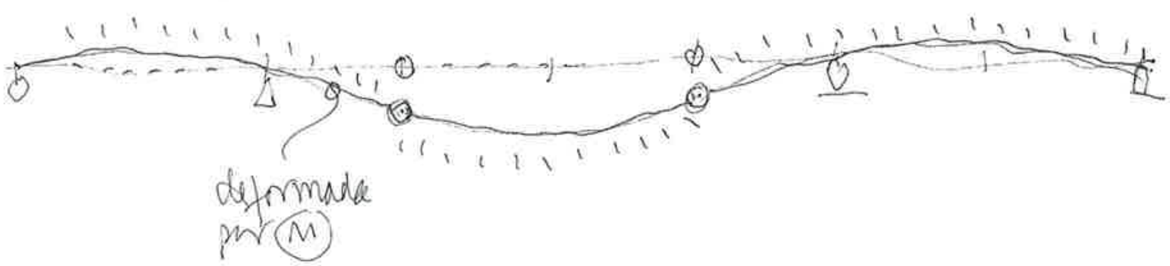
2q
2qa
D F
H=0 ⇐ ΣFx=0
3qa
3qa
a a
qa
q
3qa
A C D
0=
a a a
6qa
3qa
q
G I
F
a a a
11qa/2
qa/2



V
0 =
qa
2qa
3qa
A B C D E F G H I
=2qa
qa
3qa
qa
3qa
5qa/2
qa/2

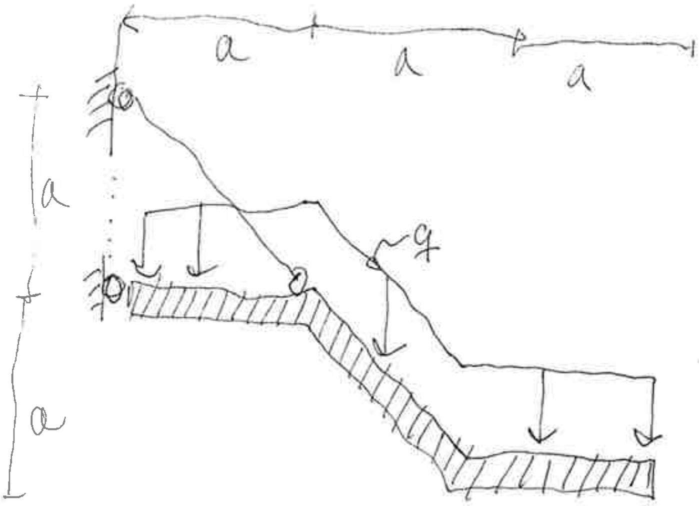


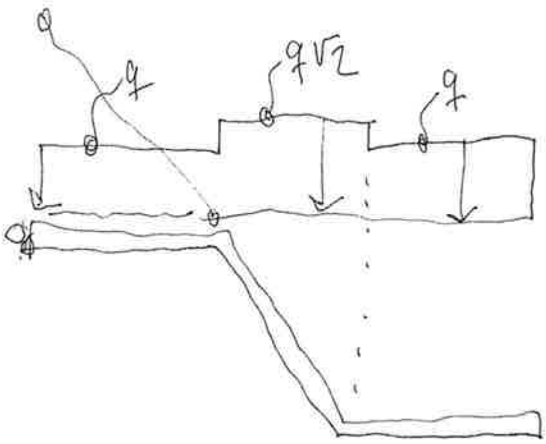
M
punto de inflexión
3qa²
lineal
qa²/2
A B C D E F
2qa²
3qa²
parábola
= q(2a)²/8 = qa²/2
qa²
= 3qa²/2 − qa²/2 = qa²
pendiente no nula en apoyo



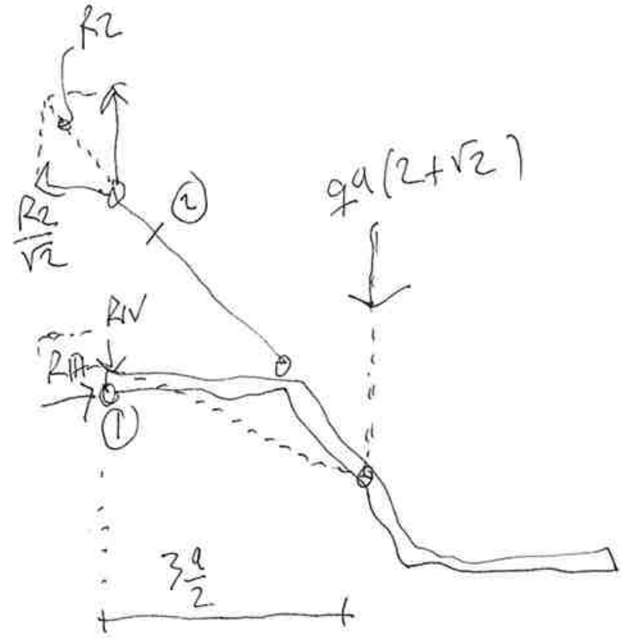
deformada por M

Marquesina



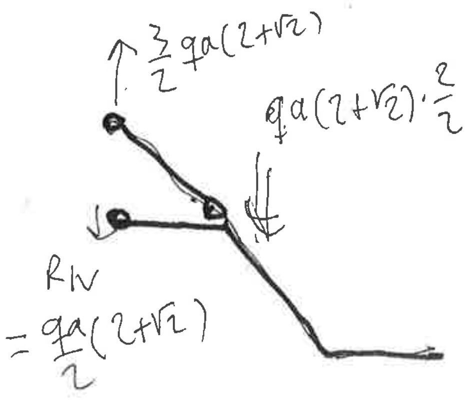


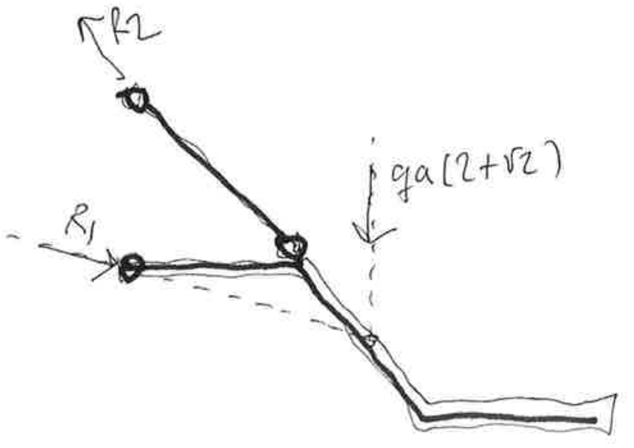
$\Rightarrow$



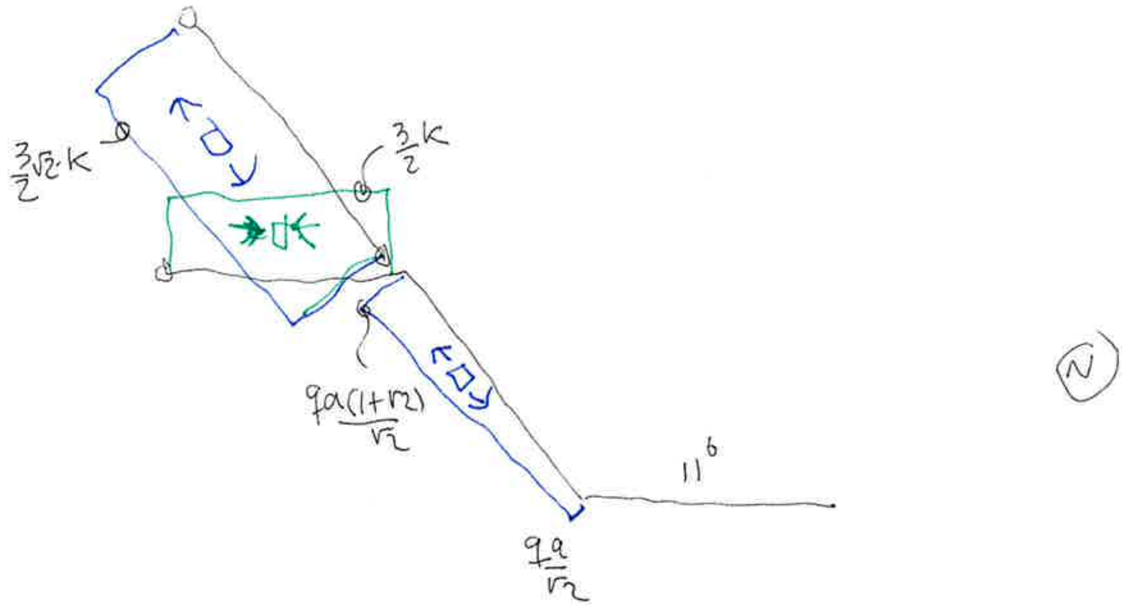
$$\sum M_{1} = 0 \Rightarrow R_2 = \frac{3\sqrt{2}}{2}(2+\sqrt{2})\,q\cdot a$$

$$R_{1H} = \frac{R_2}{\sqrt{2}} = \frac{3}{2}(2+\sqrt{2})\,q\cdot a$$

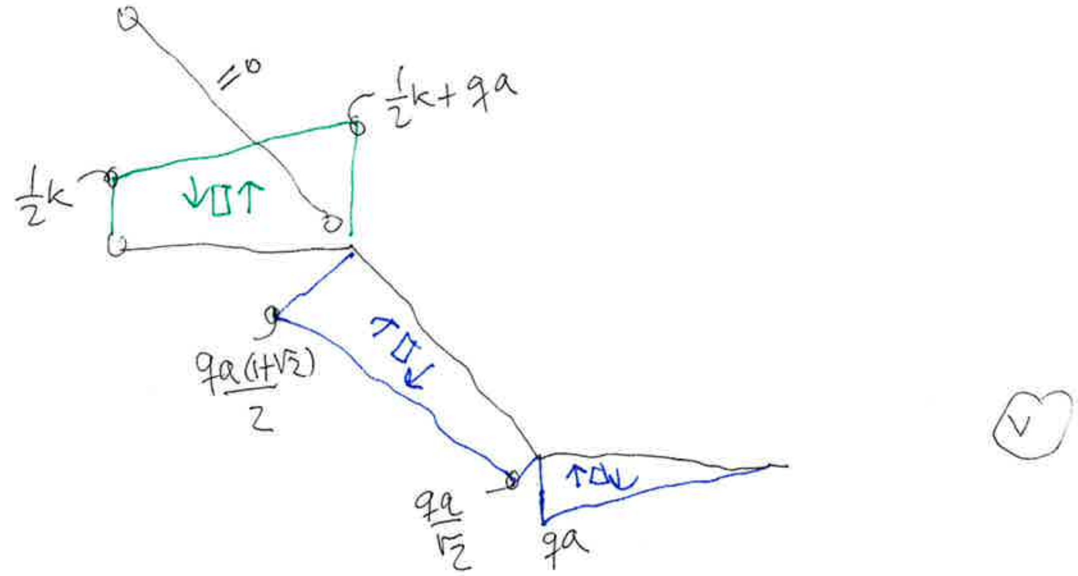




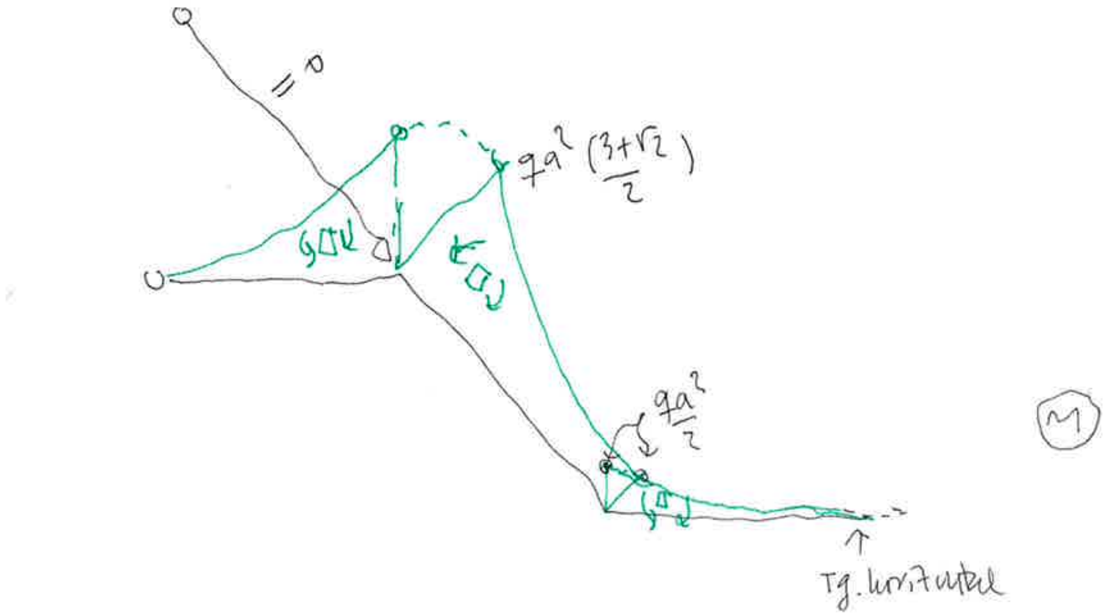
$K = (2+\sqrt{2})qa$



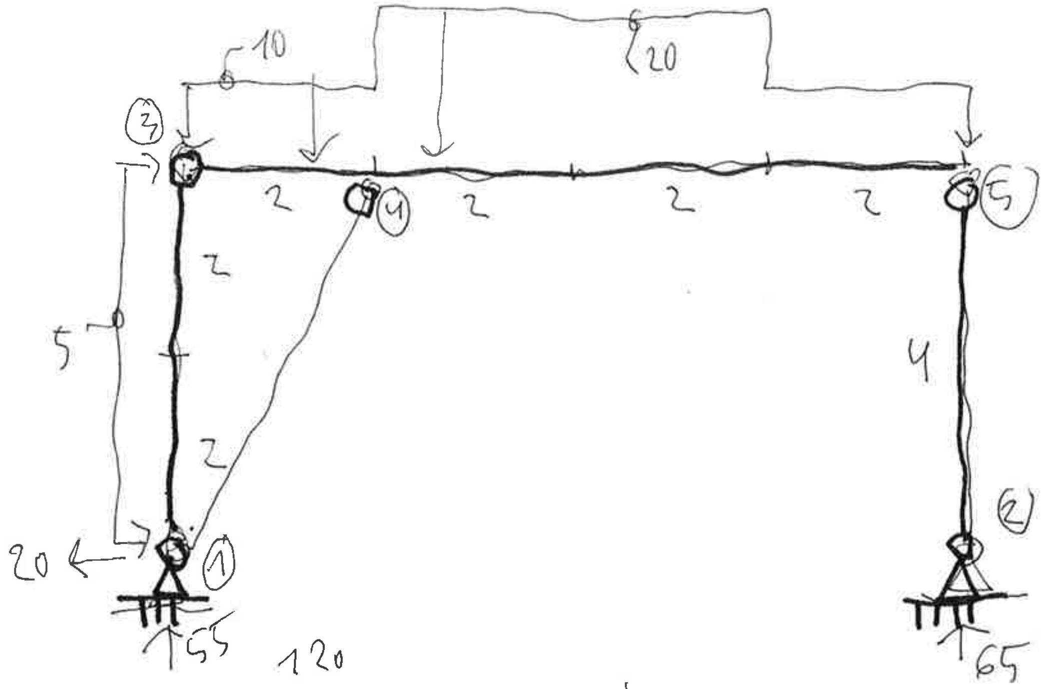
$\frac{3}{2}\sqrt{2}\cdot K$
$\frac{3}{2}K$
$\frac{qa(1+\sqrt{2})}{\sqrt{2}}$
$\frac{qa}{\sqrt{2}}$
N



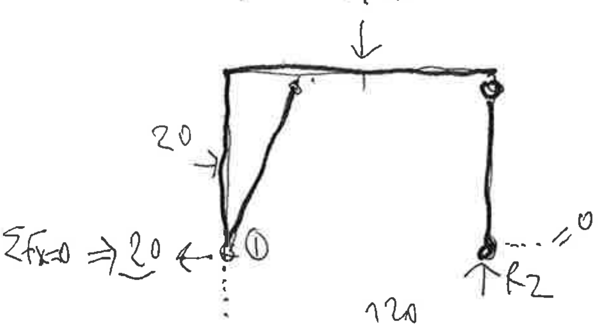
$=0$
$\frac{1}{2}K + qa$
$\frac{1}{2}K$
$\frac{qa(1+\sqrt{2})}{2}$
$\frac{qa}{\sqrt{2}}$
qa
V



$=0$
$qa^2\left(\frac{3+\sqrt{2}}{2}\right)$
$\frac{qa^2}{2}$
Tg. horizontale
M

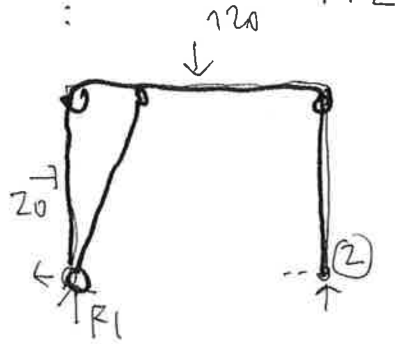


arco triarticulado

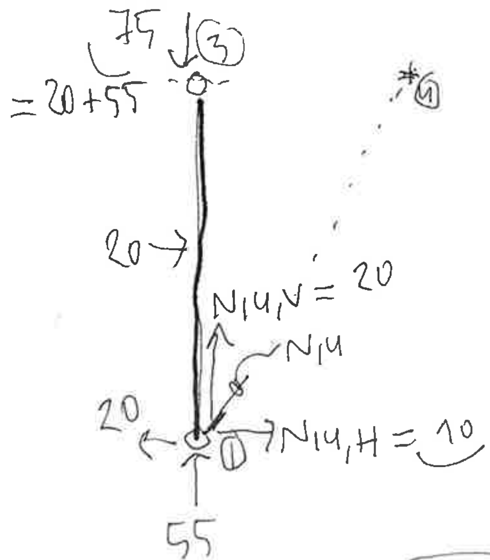


$\Sigma M_{1} = 0$

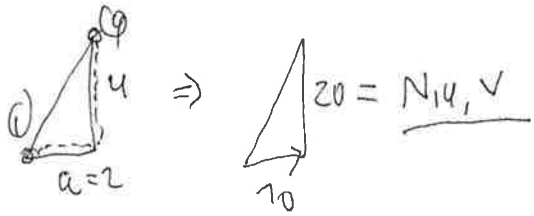
$\Rightarrow R_2 = 65$

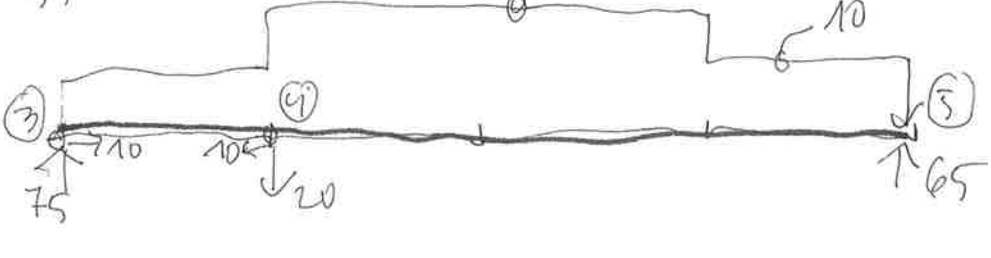


$\Sigma M_{2} = 0 \Rightarrow R_1 = 55$

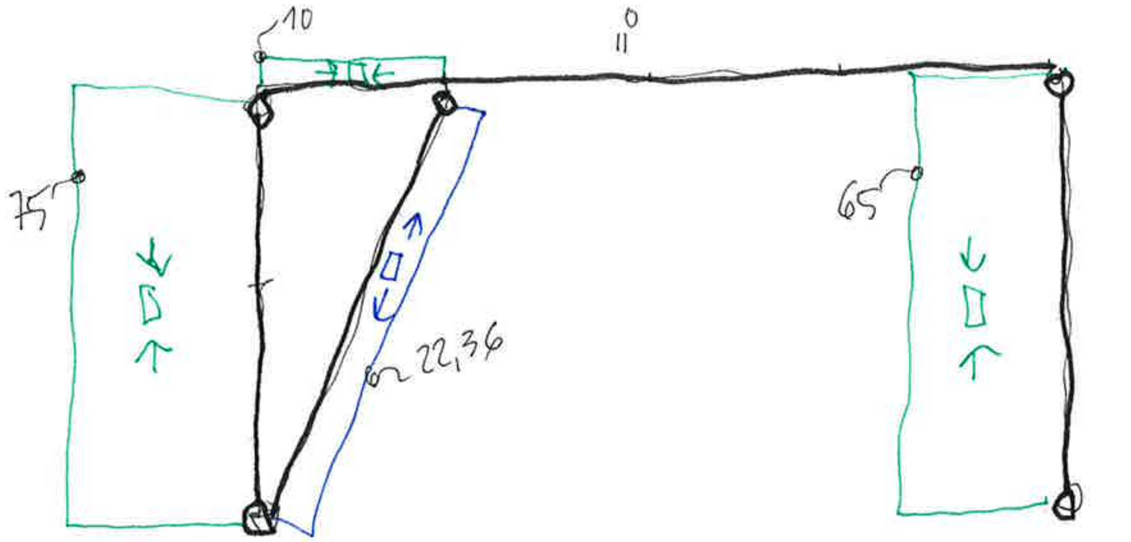


$\Sigma M_{3} = 0 \Rightarrow N_{14,H} = 10$

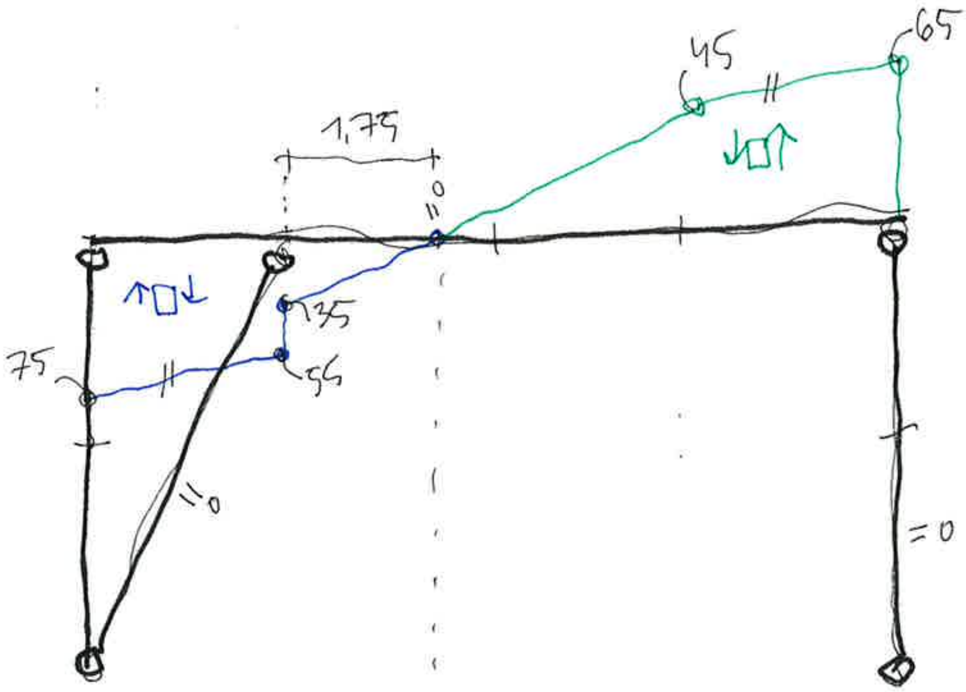




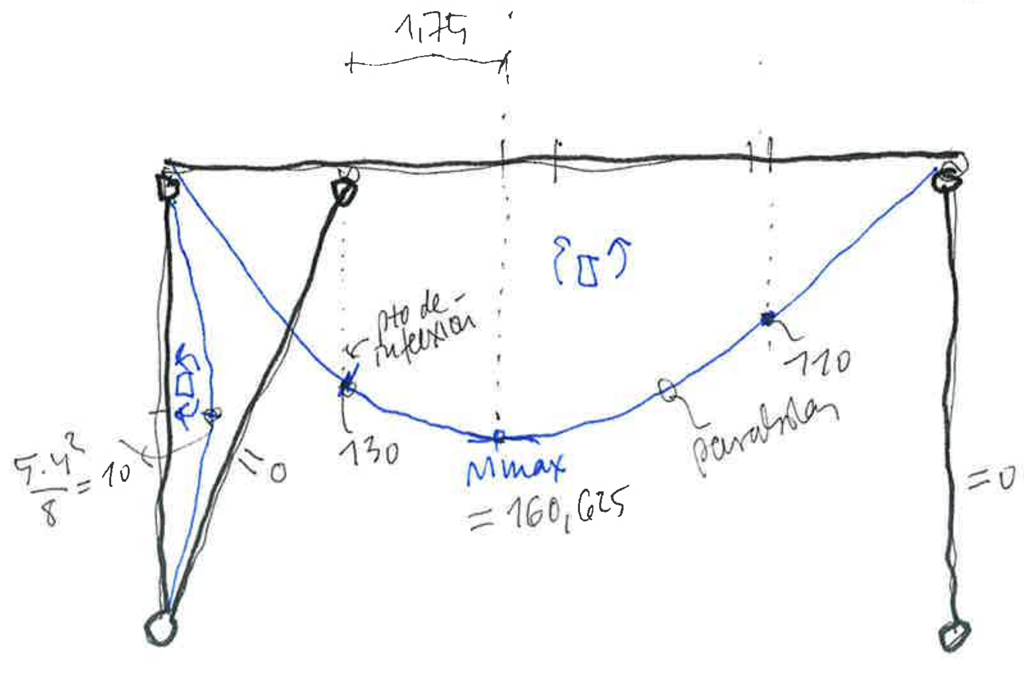
equilibrio en el dintel superior



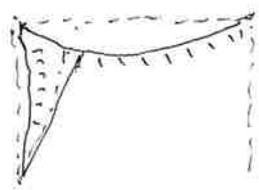
10
=0
75
65
22,36
N

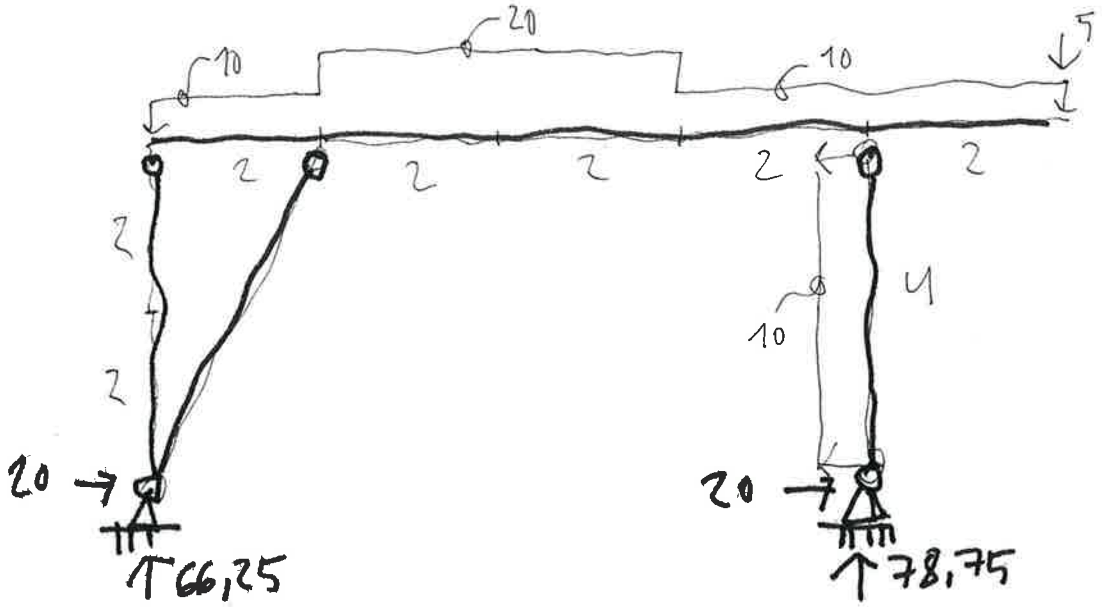


1,75
=0
45
65
135
55
75
=0
=0
V

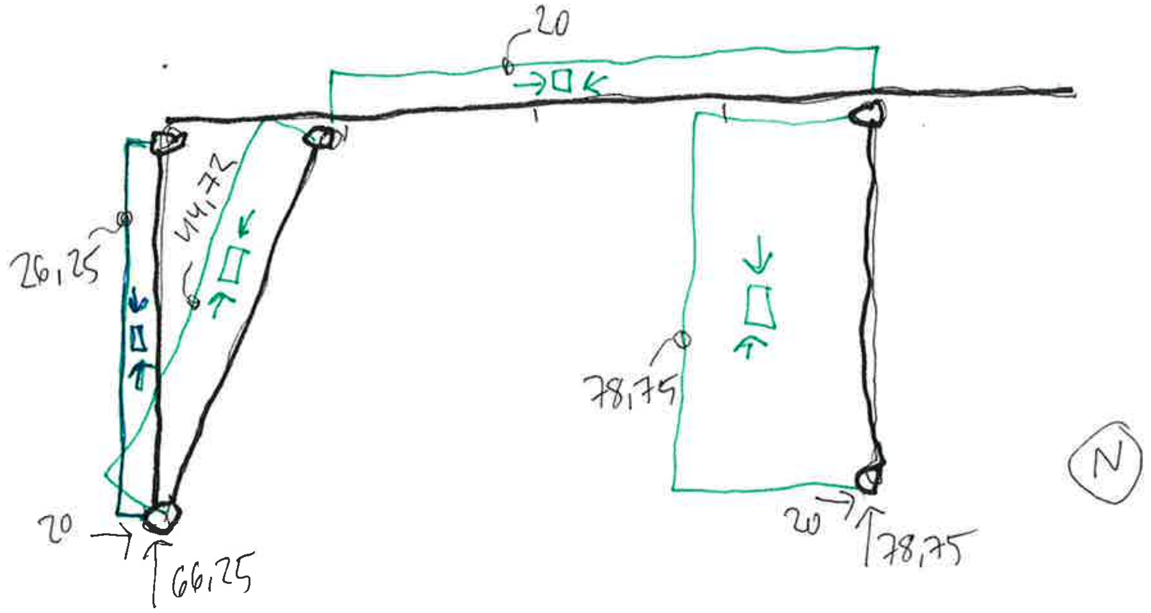


1,75
pto de inflexion
110
130
parabola
$\frac{5 \cdot 4^2}{8} = 10$
=0
=0
Mmax
= 160,625
M

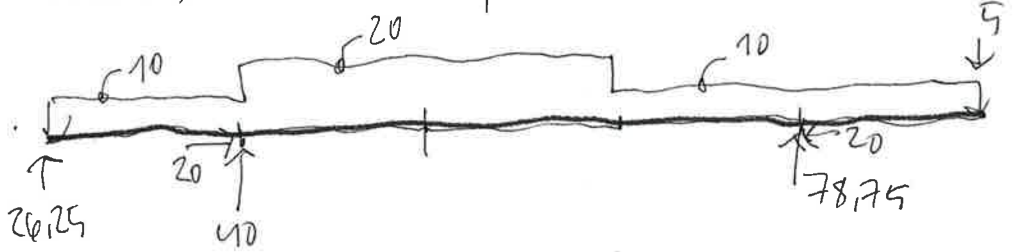


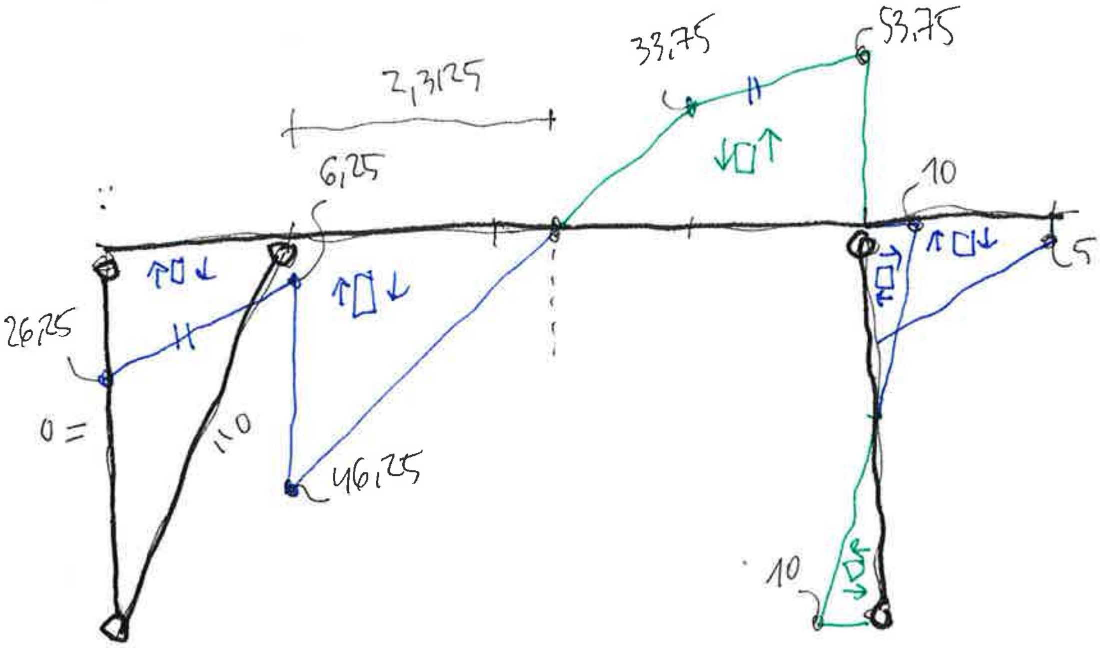


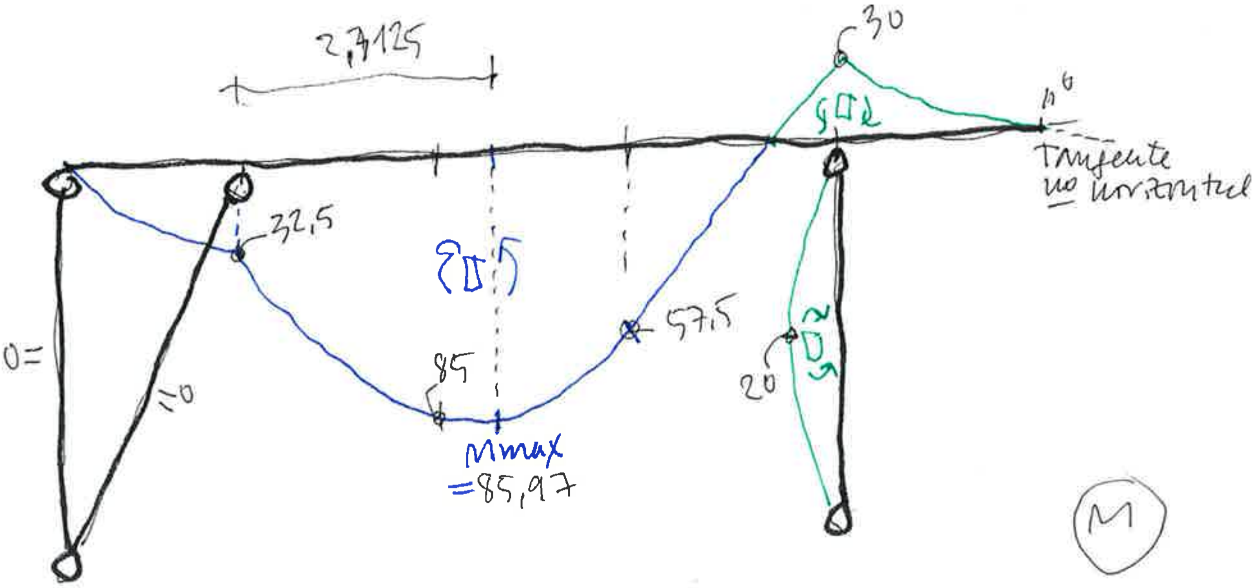
arco triarticulado.



el método para obtener reacciones y normales es idéntico al caso anterior analisis del dintel superior







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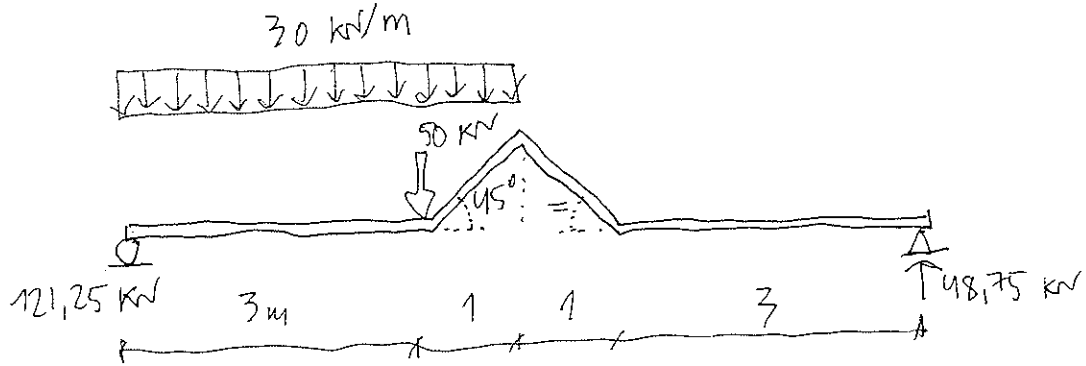
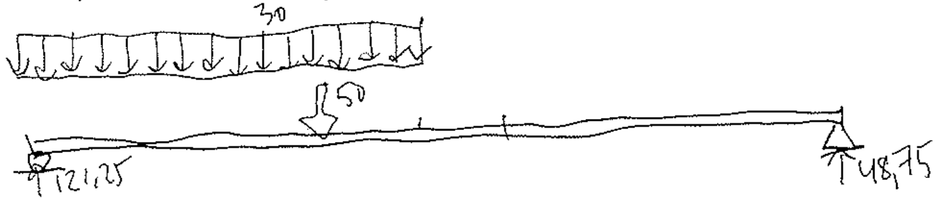
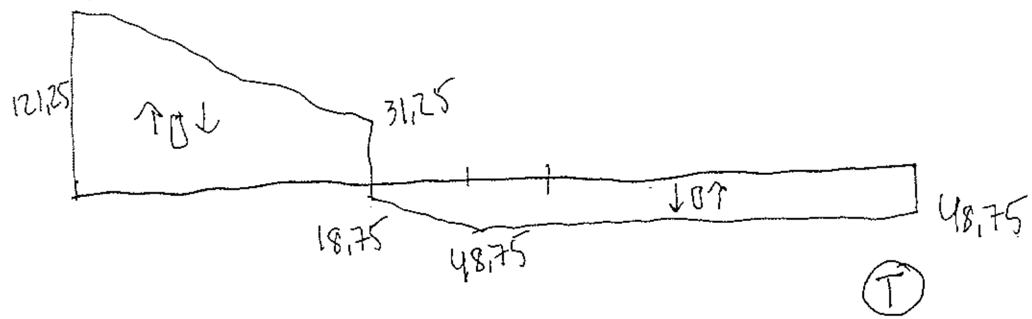
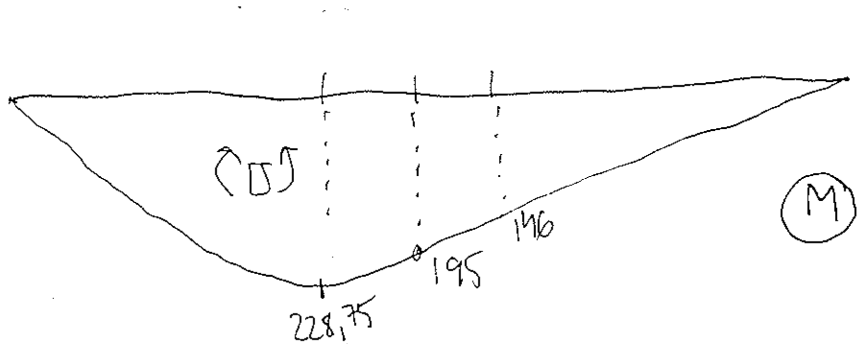
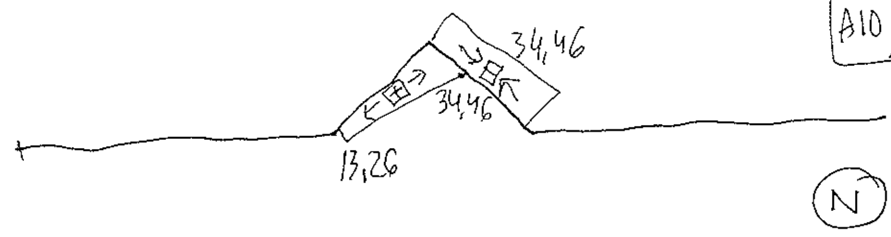


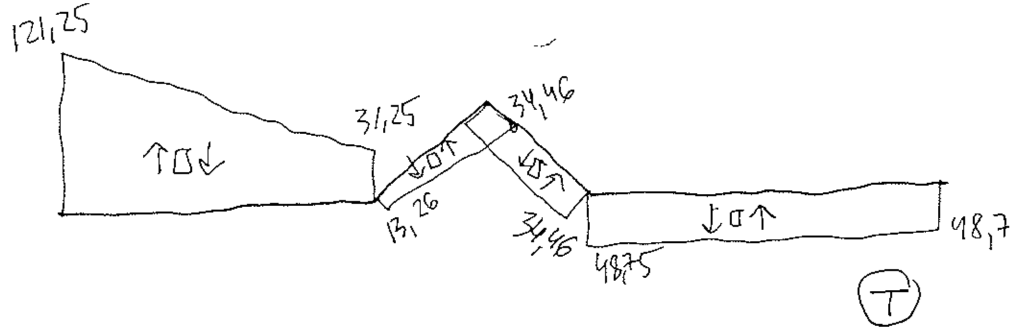
diagrama de la viga proyectada de referencia.

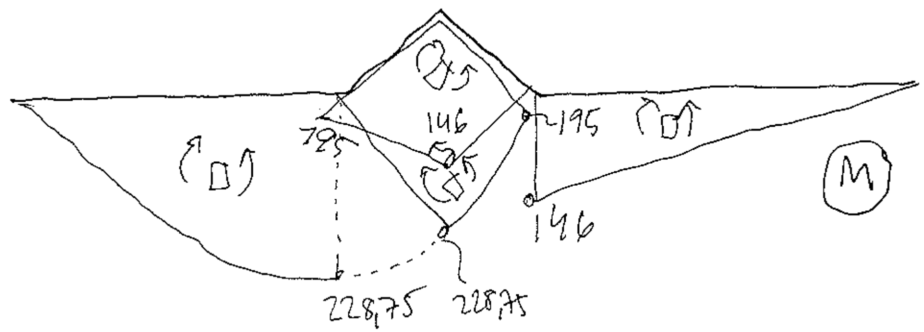




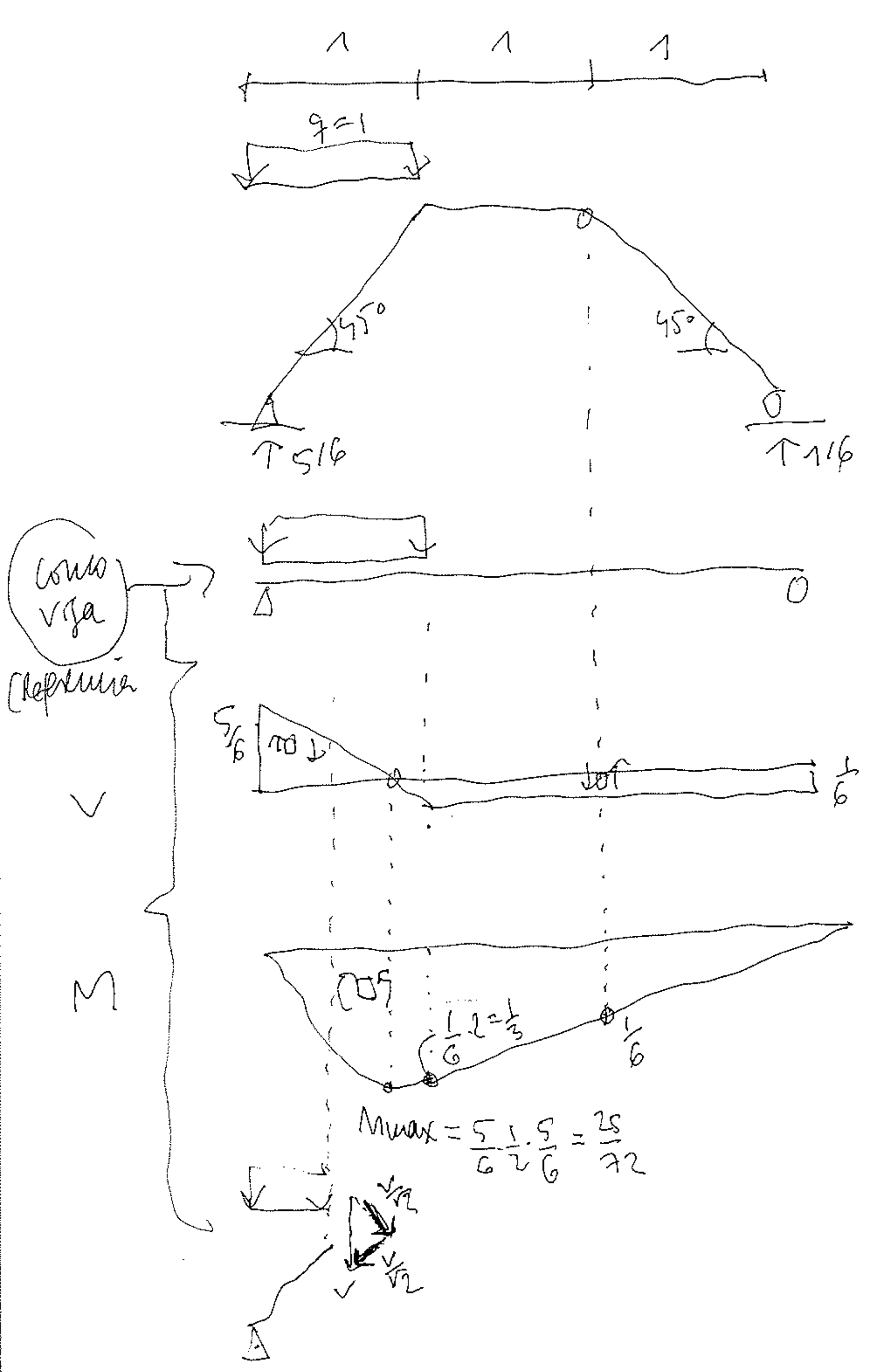




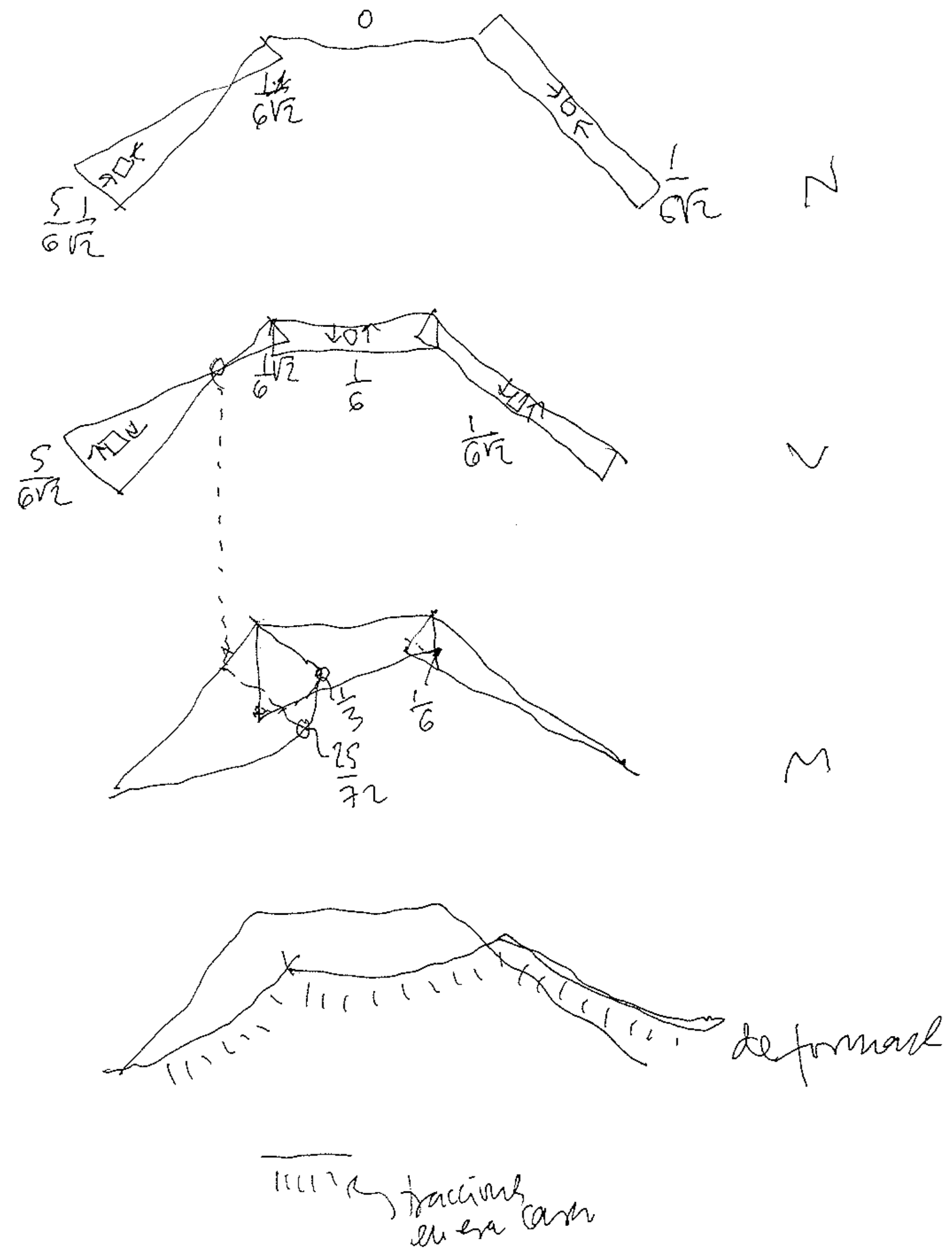




Δ 11/60



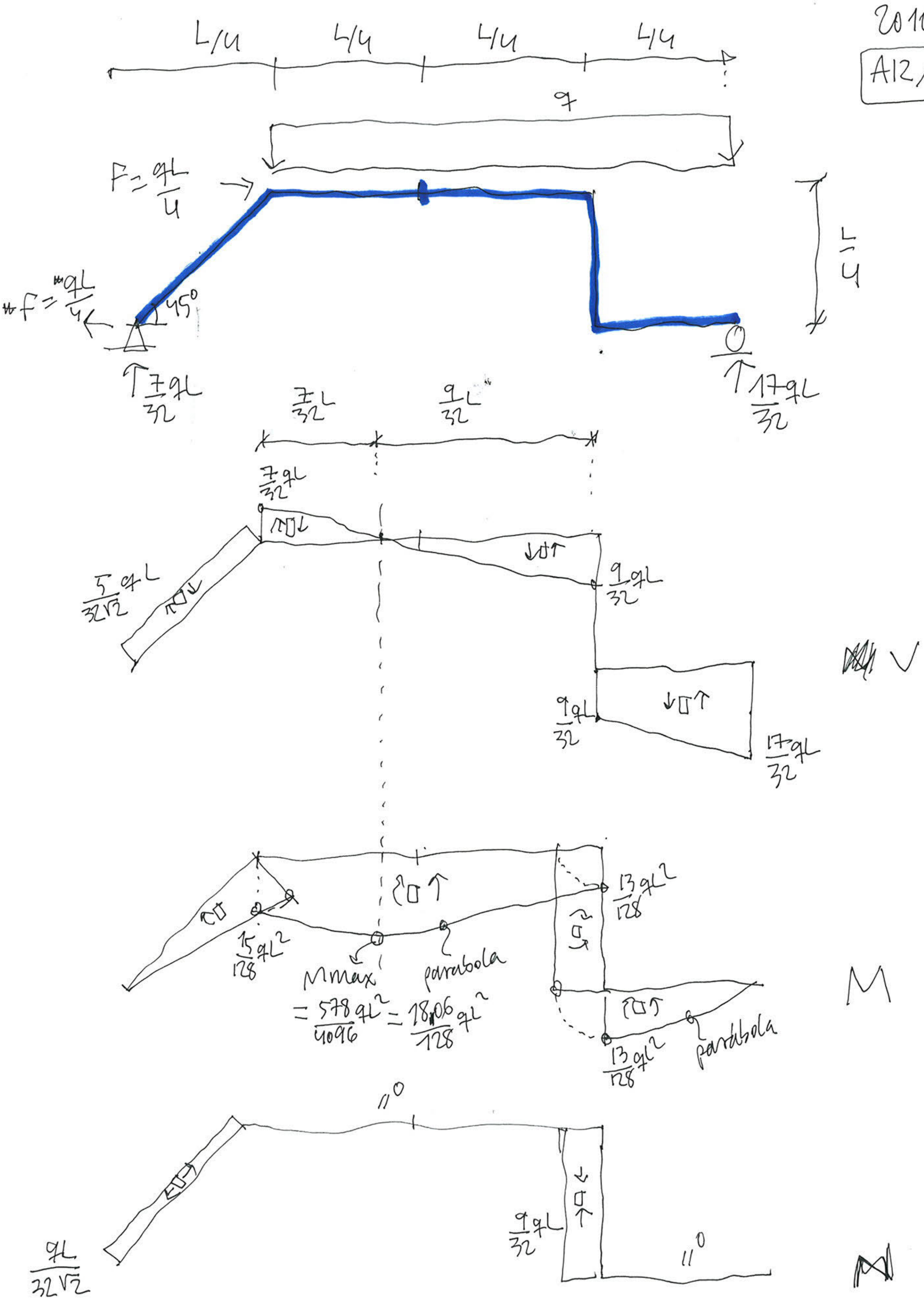
1 1 1
q=1
45°
45°
↑5/6
↑1/6
0
como viga
equivalente
V
M
5/6
1/6
$\frac{1}{6} \cdot 2 = \frac{1}{3}$
$\frac{1}{6}$
$M_{max} = \frac{5}{6} \cdot \frac{1}{2} \cdot \frac{5}{6} = \frac{25}{72}$
$\frac{V}{\sqrt{2}}$
$\frac{V}{\sqrt{2}}$
V



N
0
$\frac{1}{6\sqrt{2}}$
$\frac{5}{6\sqrt{2}}$
$\frac{1}{6\sqrt{2}}$
V
$\frac{5}{6\sqrt{2}}$
$\frac{1}{6\sqrt{2}}$
$\frac{1}{6}$
$\frac{1}{6\sqrt{2}}$
M
$\frac{1}{3}$
$\frac{25}{72}$
$\frac{1}{6}$
deformada
tracciones en esa cara

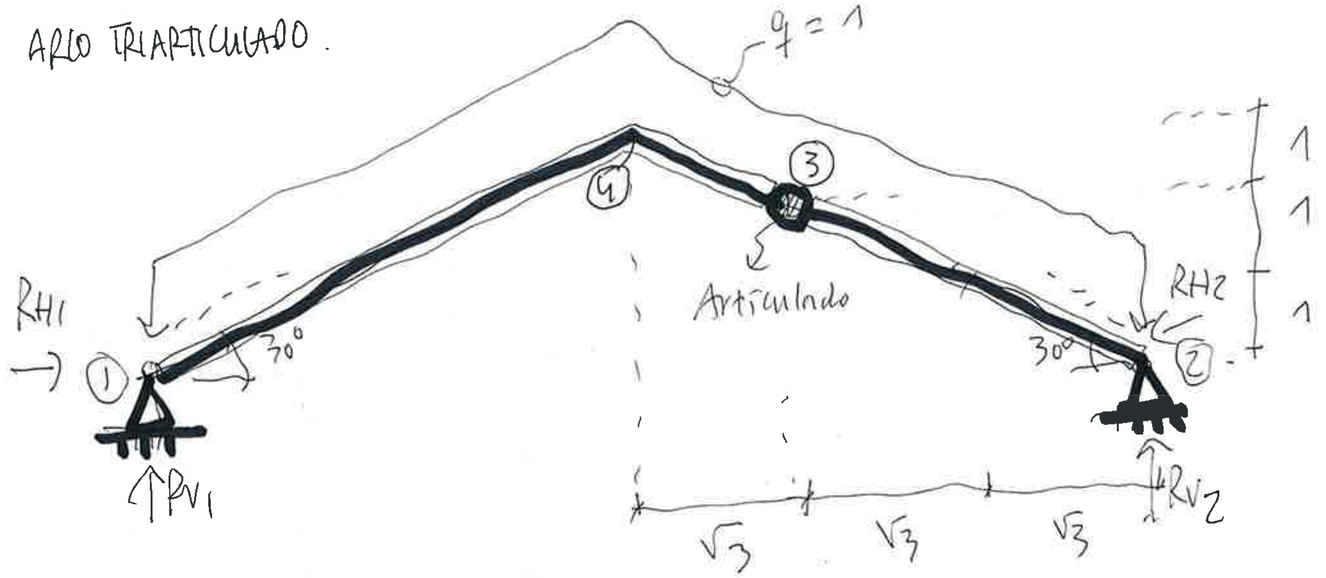
2010

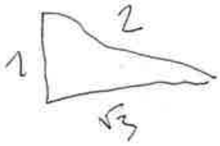
A12/10



L/4
L/4
L/4
L/4
q
F = qL/4
L/4
f = qL/4
45°
7/32 qL
17/32 qL
7/32 L
9/32 L
7/32 qL
5/(32√2) qL
9/32 qL
9/32 qL
17/32 qL
V
13/128 qL²
15/128 qL²
Mmax = 578/4096 qL² = 18,06/128 qL²
parabola
13/128 qL²
parabola
M
9/(32√2) qL
9/32 qL
N

ARCO TRIARTICULADO.





||

resultante acciones.. ⇩ $P = \overset{q}{1} \cdot \overset{L}{6} \cdot 2 = 12$

$RV_1 \uparrow = \frac{12}{2} = 6$

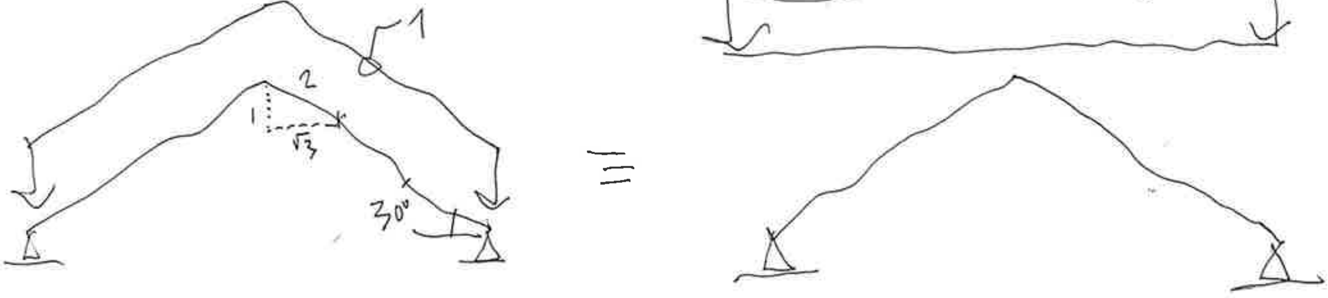
las reacciones verticales son independientes de empuje

$\left\{ \uparrow RV_2 = \frac{12}{2} = 6 \right.$

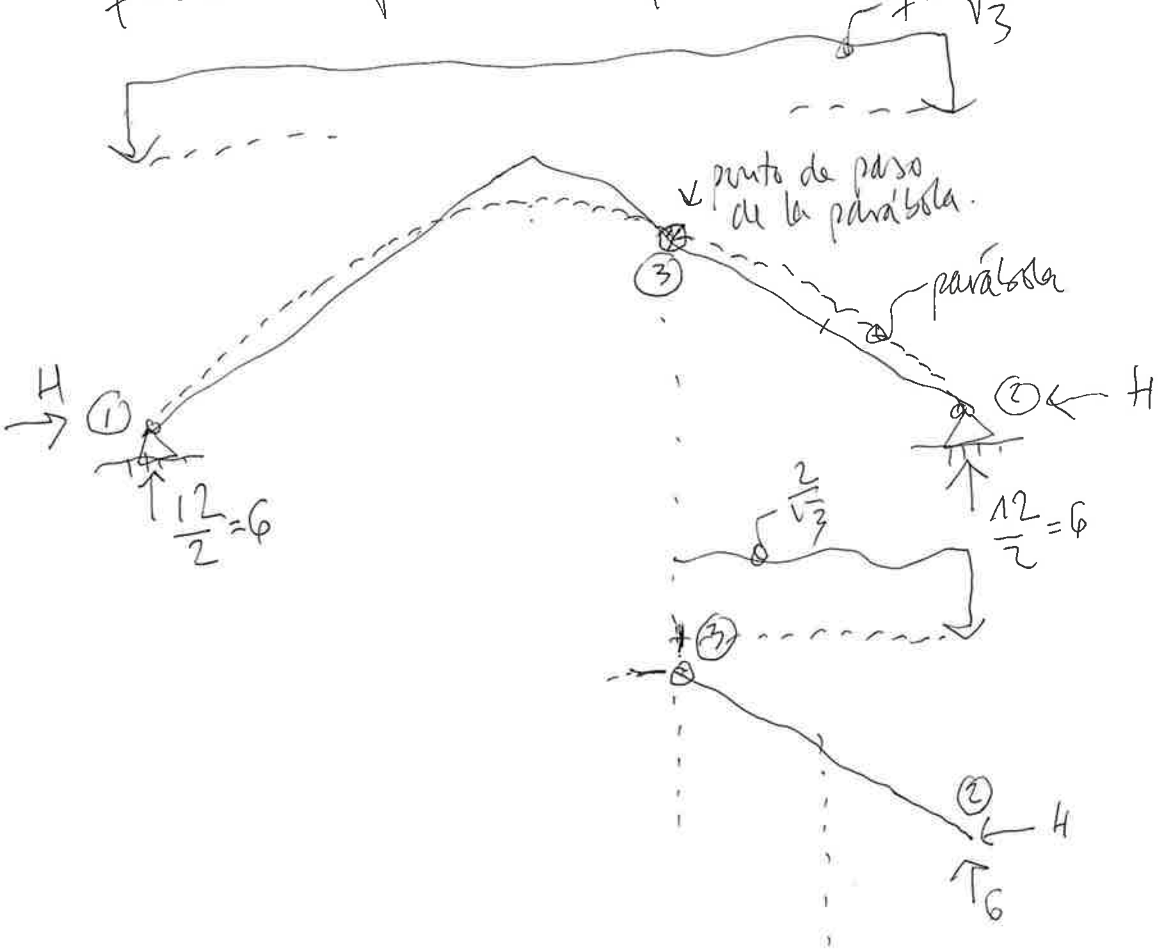
en general, considerando H = H' según la dirección que une los arranques, R_{V_1} y R_{V_2} son también independientes del empuje H'.

Si se proyecta la carga horizontalmente se visualiza mejor la situación.

$$1\cdot\frac{1}{\cos 30} = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}} = q'$$



$q = cte \rightarrow$ funicular = parábola.

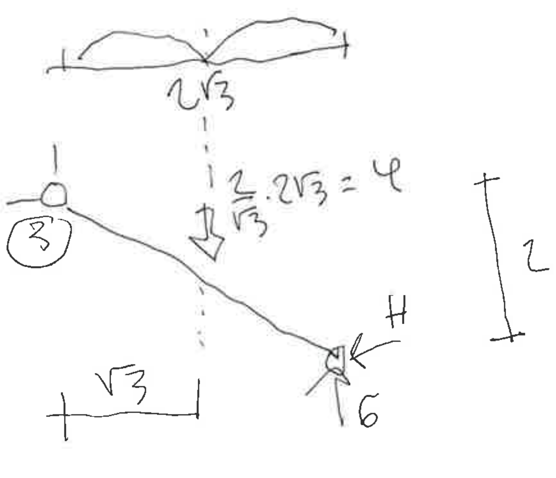


$\Sigma M_{3} = 0$ $+\circlearrowleft$

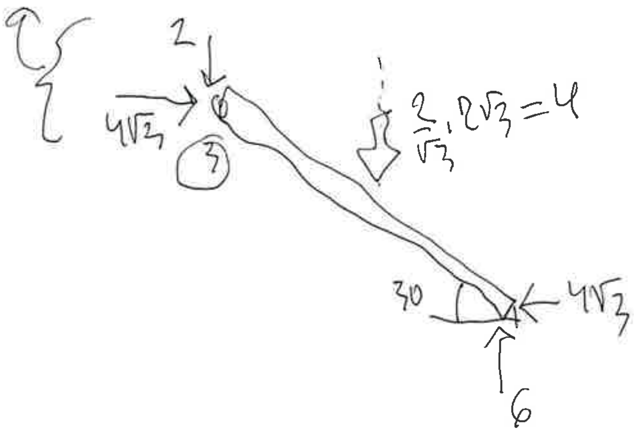
$H\cdot 2 + 4\cdot\sqrt{3} - 6\cdot 2\sqrt{3} = 0$

$\underline{H} = \frac{8\sqrt{3}}{2} = \underline{4\sqrt{3}}$

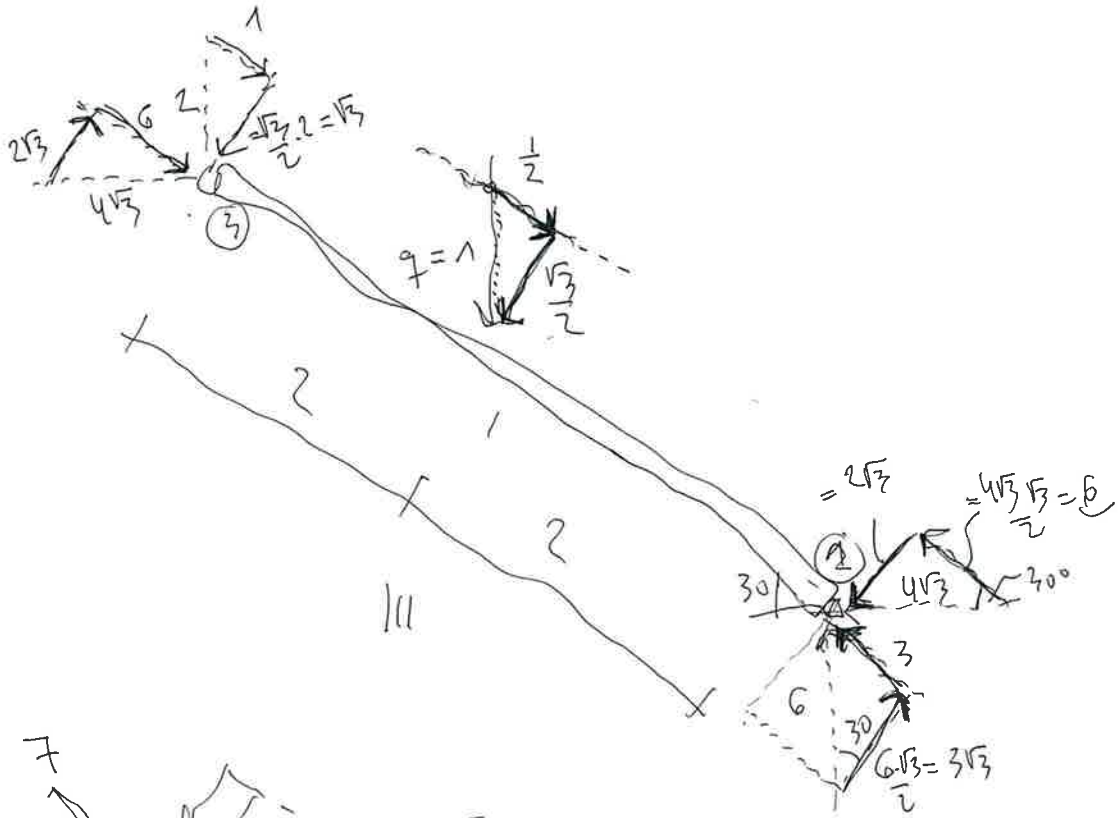
≡

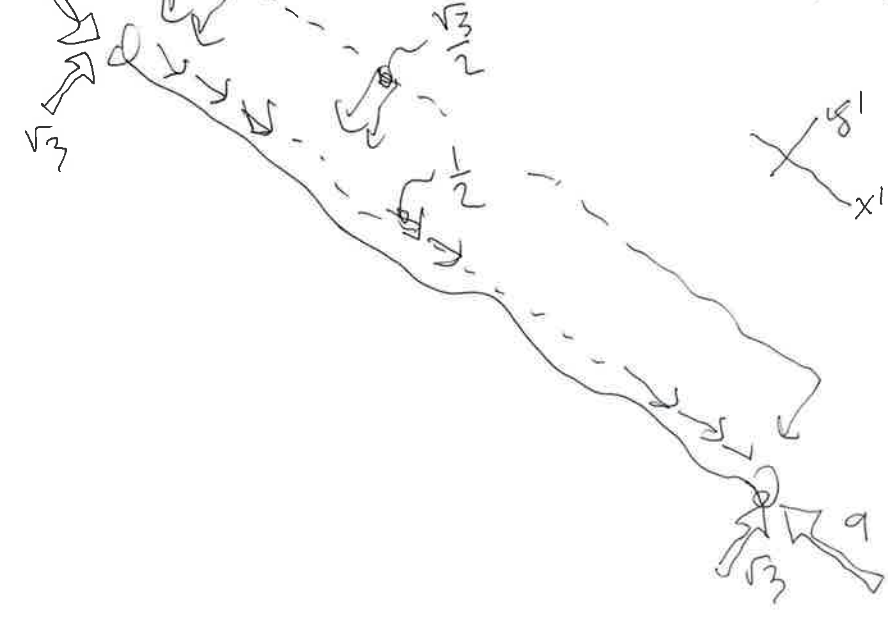


esfuerzos en ③



descomponemos según la dirección de la barra.

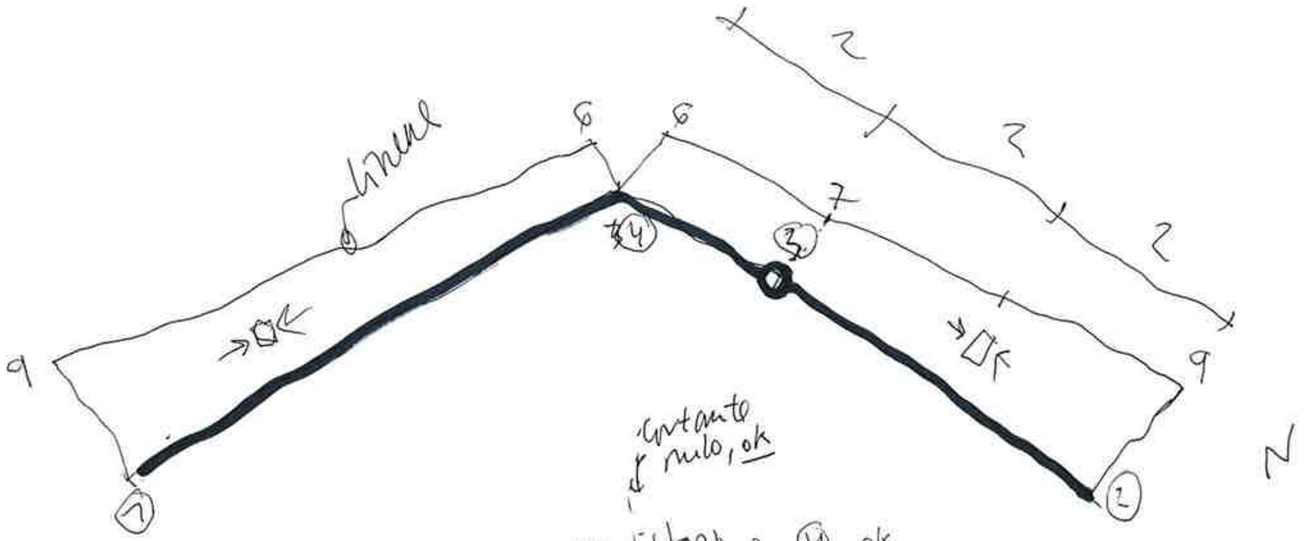




$$\left\{\begin{array}{l} \Sigma F_{x'} = 0 \\ 7 + \frac{1}{2} \cdot 4 \underset{ok}{=} 9 \end{array}\right.$$

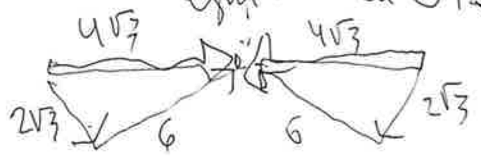
$$\left\{\begin{array}{l} \Sigma F_{y'} = 0 \\ \frac{\sqrt{3}}{2} \cdot 2 \underset{ok}{=} \sqrt{3} \end{array}\right.$$

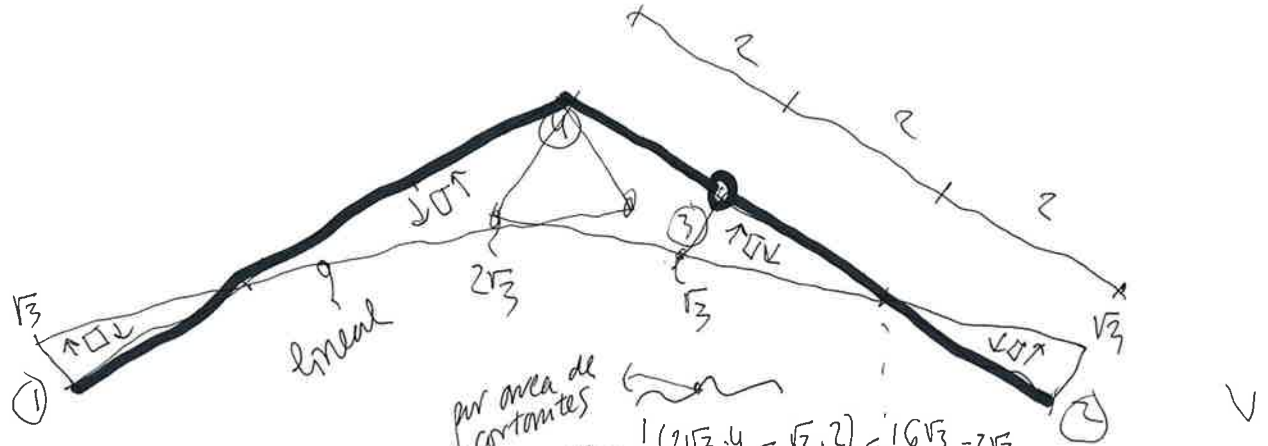
"cierre"



cortante nulo, ok

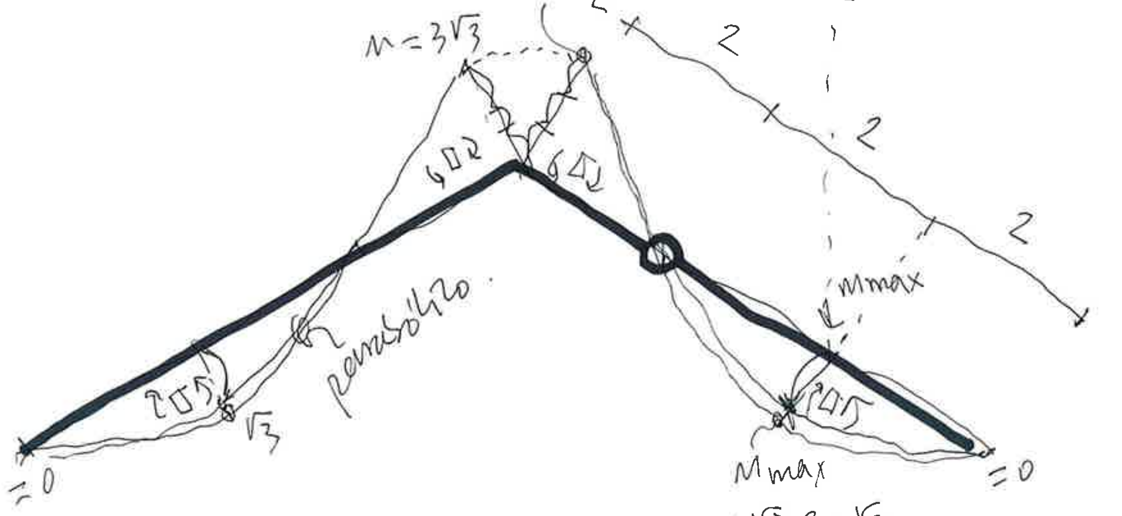
equilibrio en ④, ok





por area de cortantes

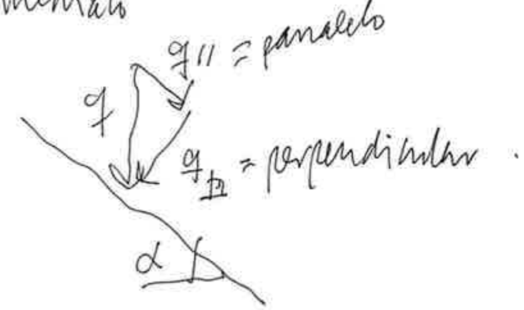
$$M=\frac{1}{2}\left(2\sqrt{3}\cdot 4-\sqrt{3}\cdot 2\right)=\frac{1}{2}6\sqrt{3}=3\sqrt{3}$$

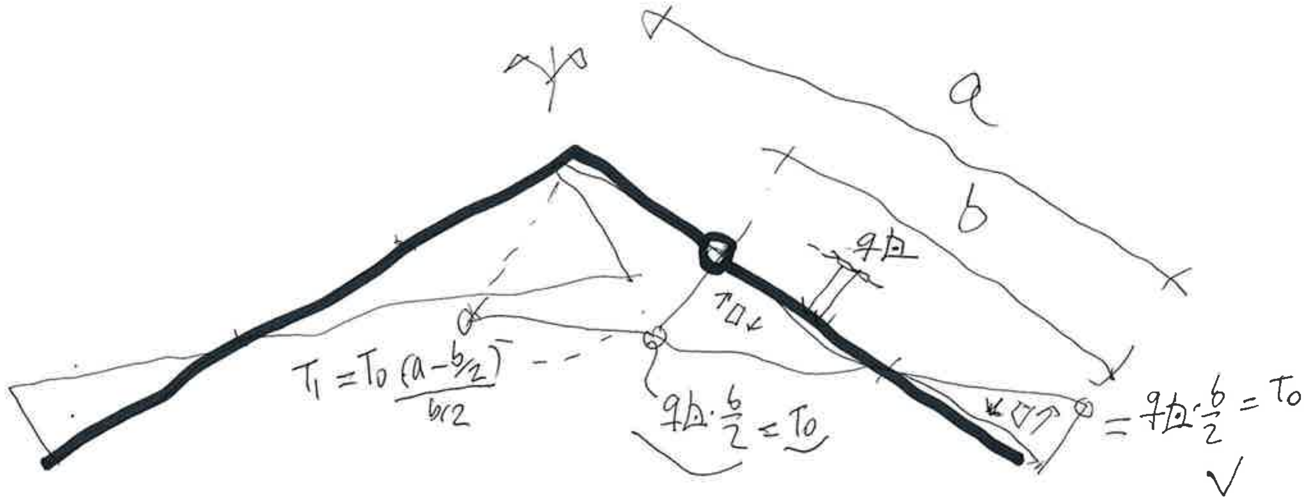


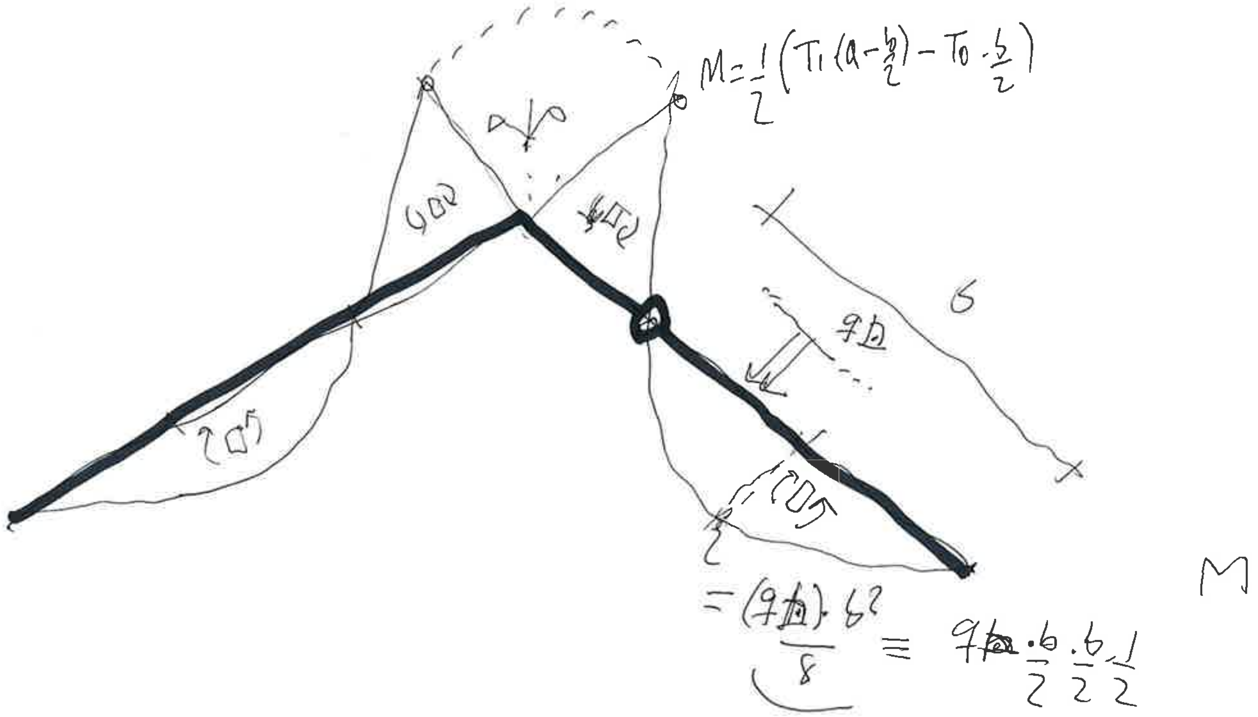
Mmax

$$=\frac{\sqrt{3}\cdot 2}{2}=\sqrt{3}$$

dado la simetría, el proceso para obtener los diagramas es entonces inmediato



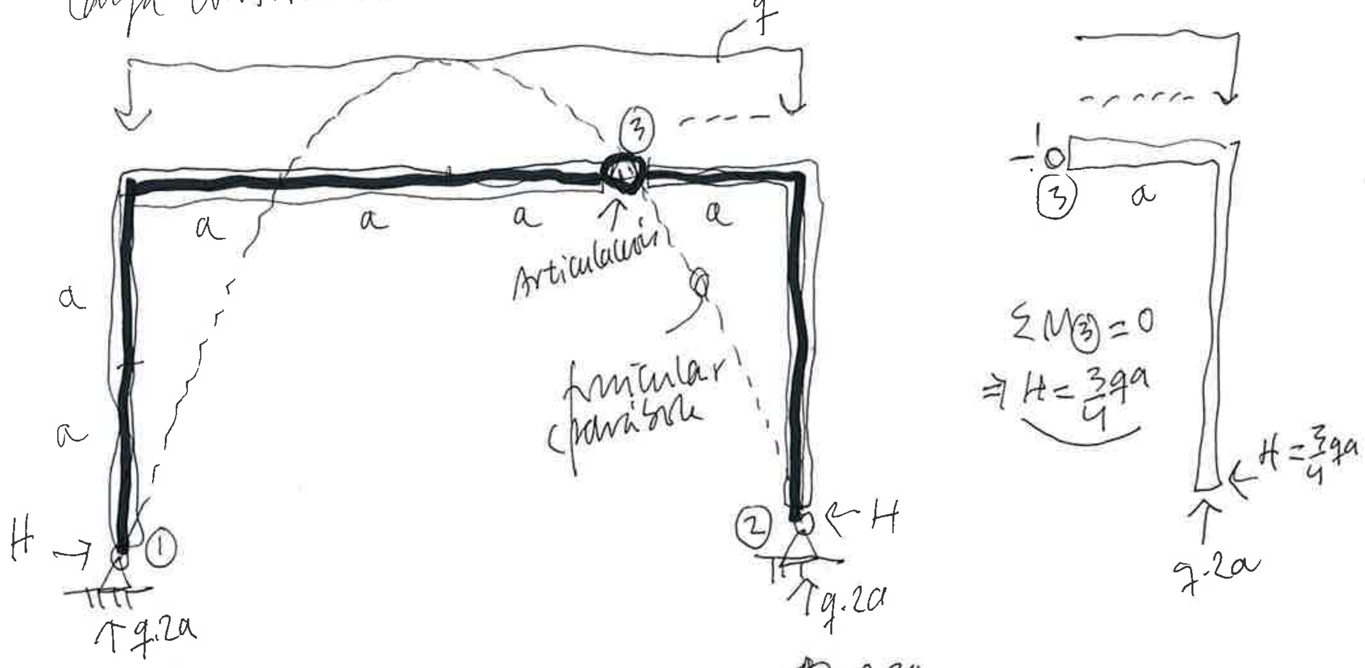


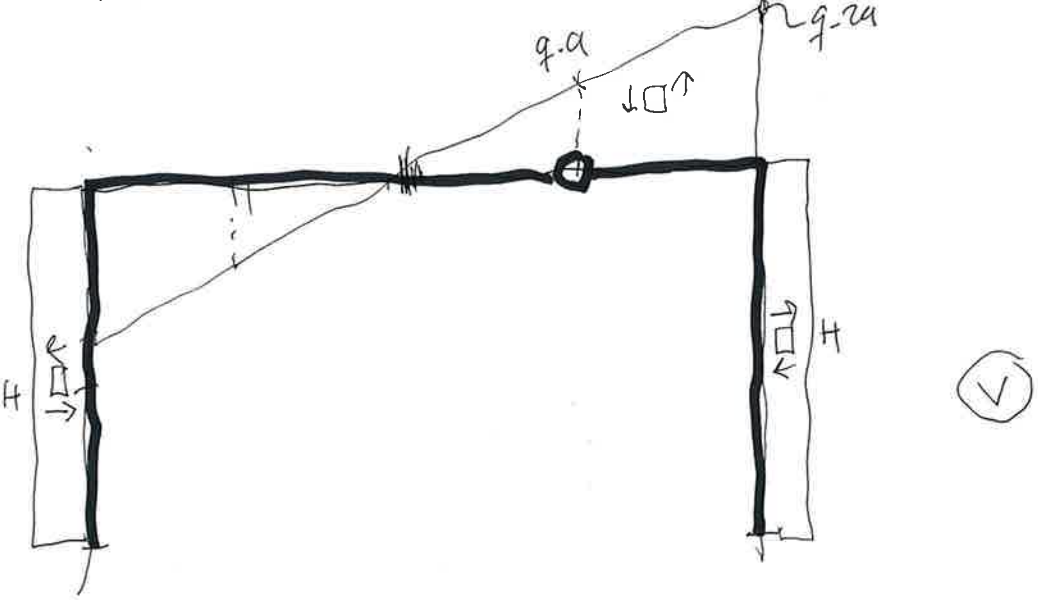


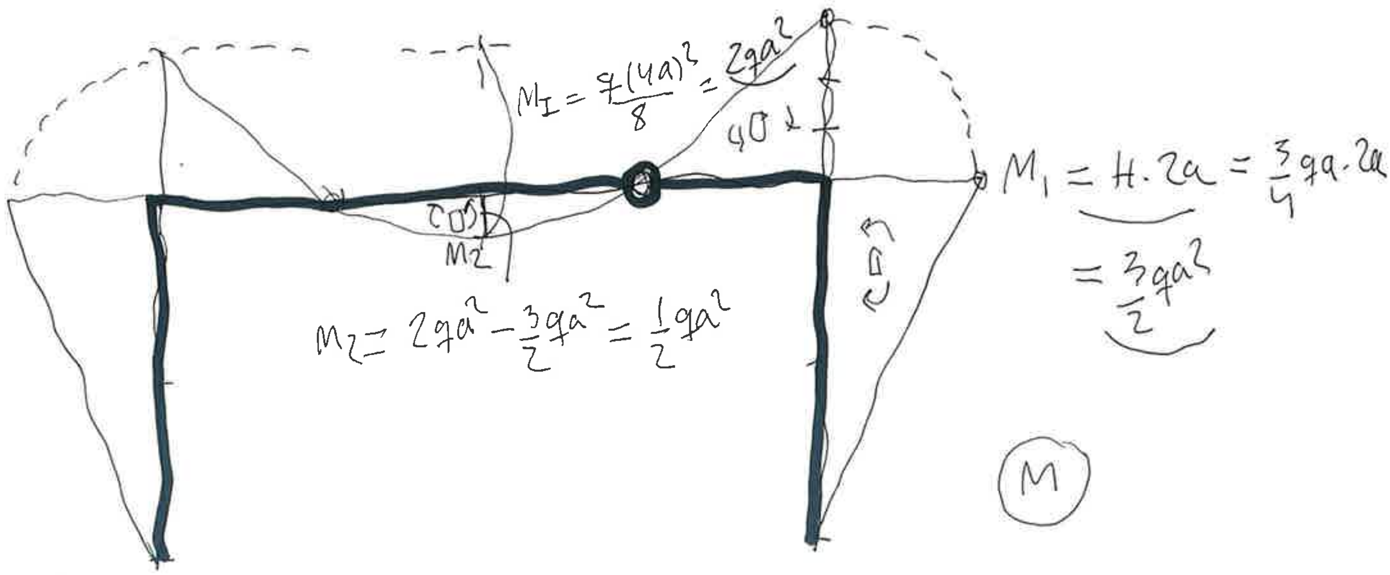
y por tanto V y M
son función de $q_{\perp}$ y de la
posición de la rótula.

el normal depende del empuje que a su vez
depende del peralte (canto) del arco.

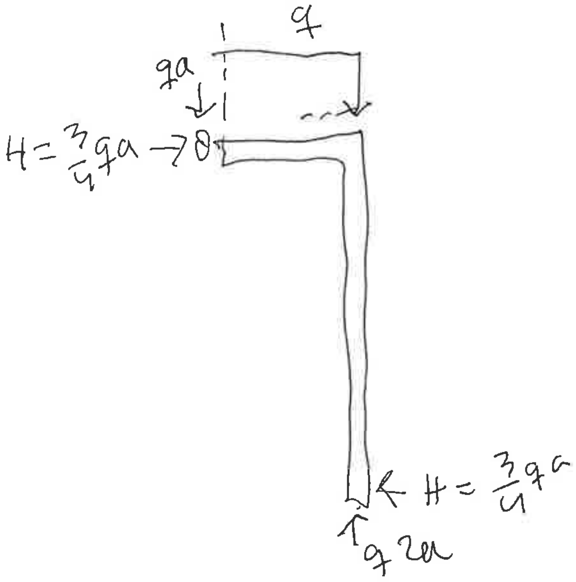
... de forma similar, con otra geometría, pero también un ARCO TRIARTICULADO y una carga continua.

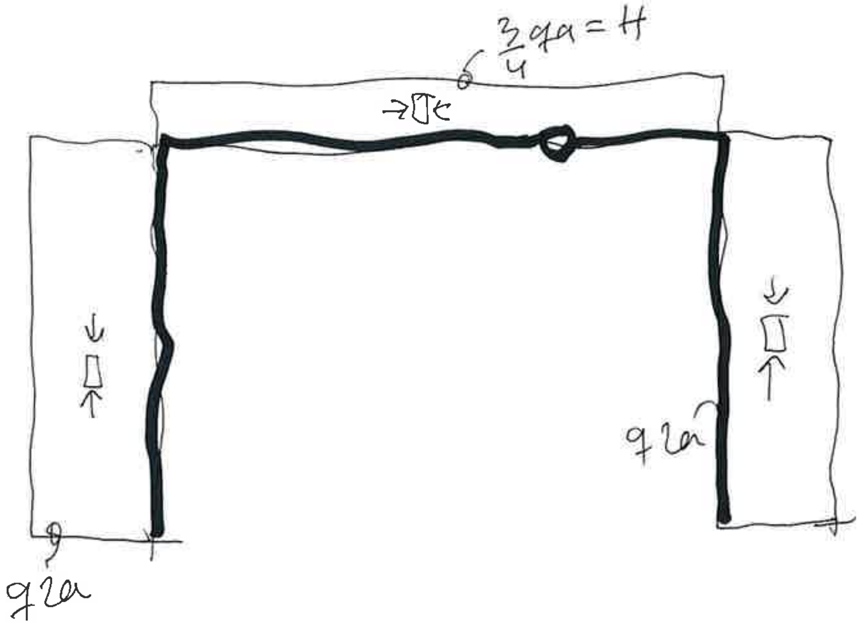






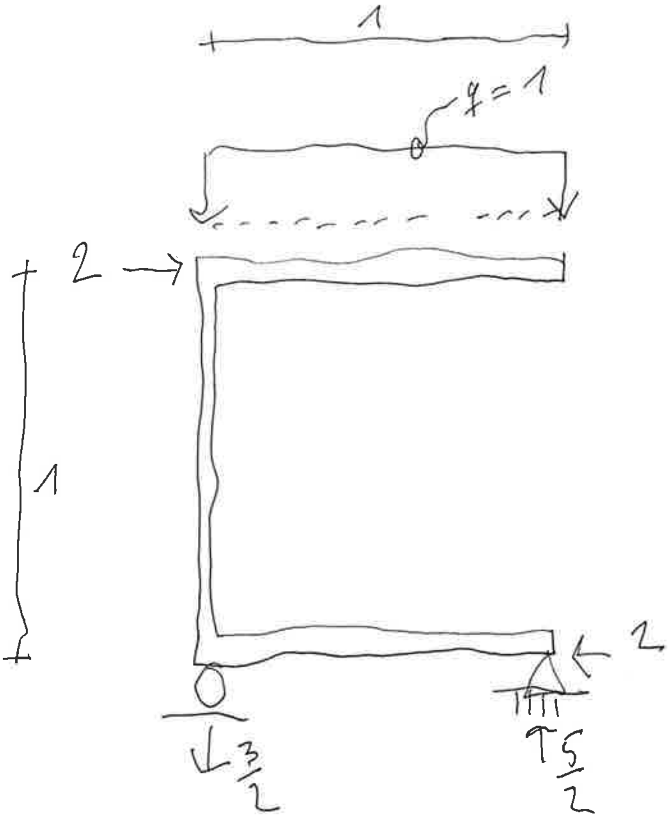
... normales

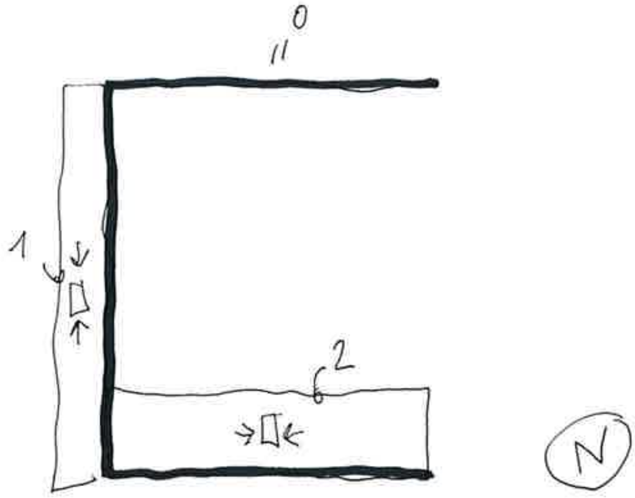


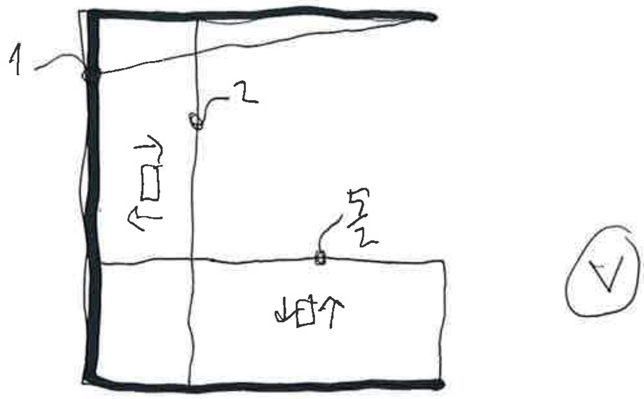


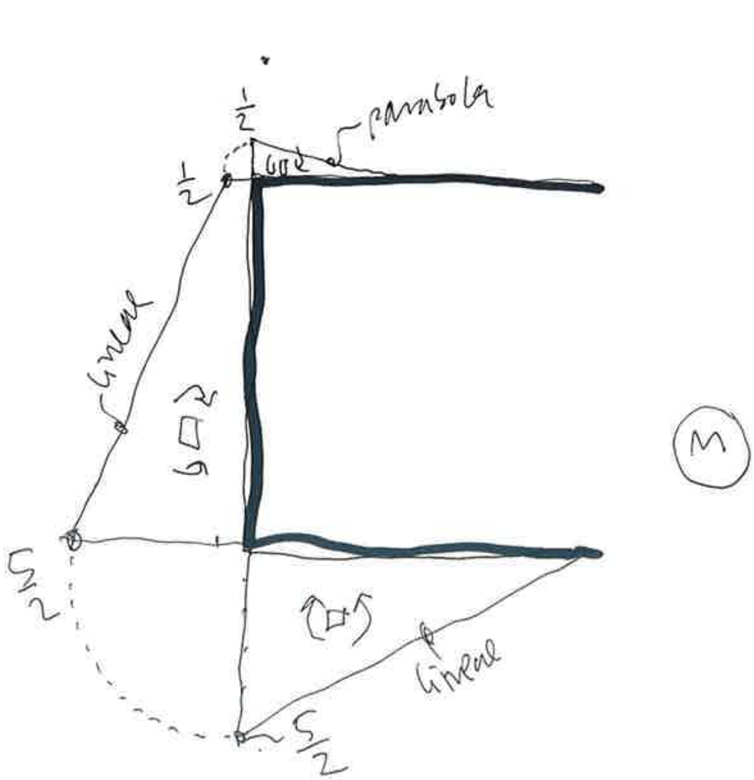
N

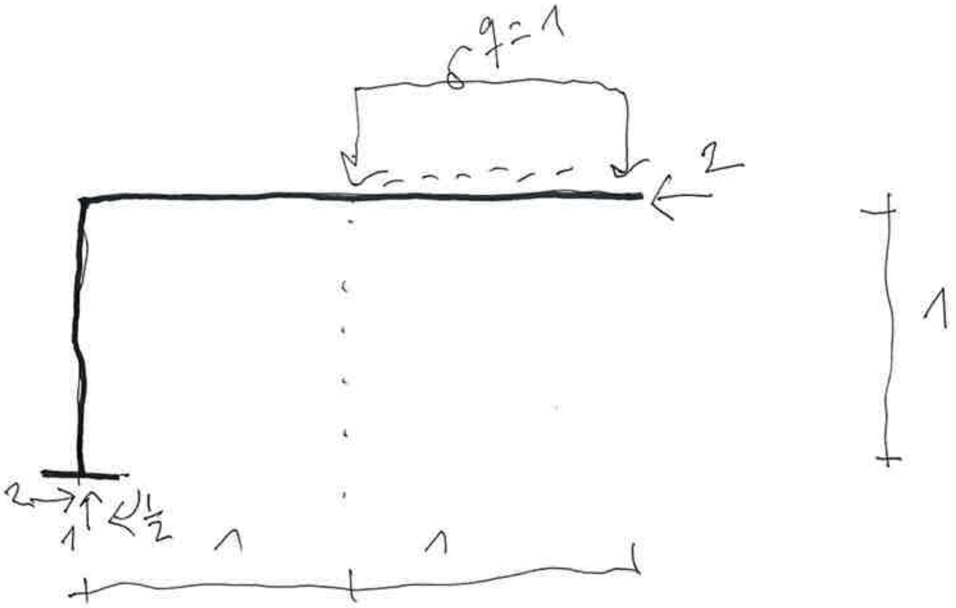
1

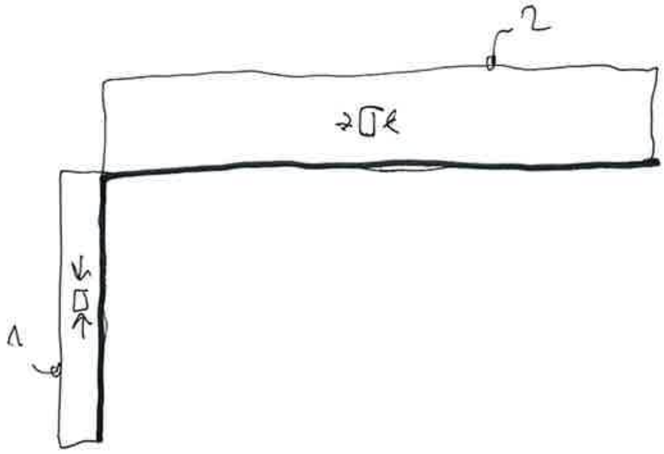

1
q = 1
2
1
2
3/2
5/2

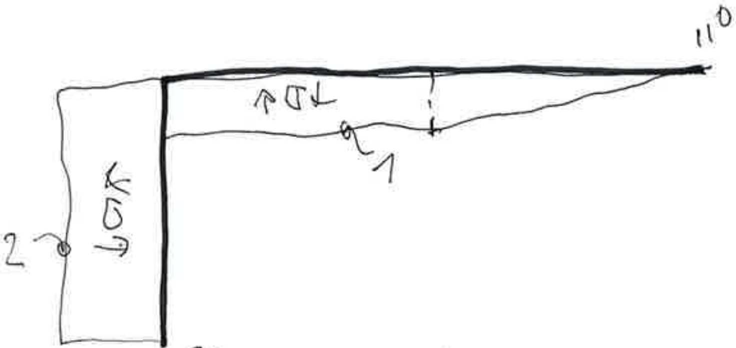

0
1
2
N

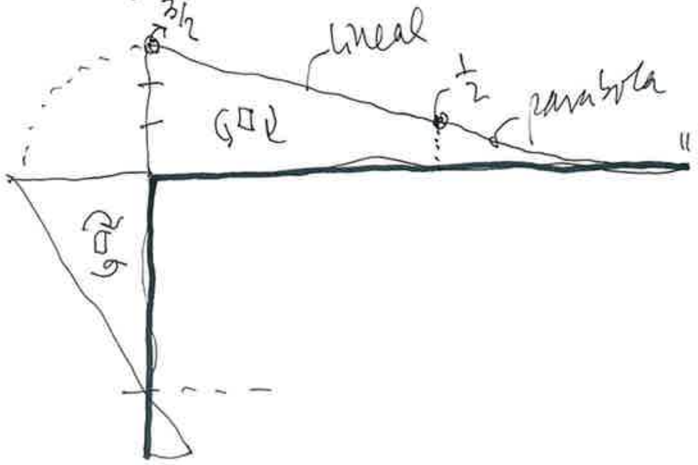

1
2
5/2
V

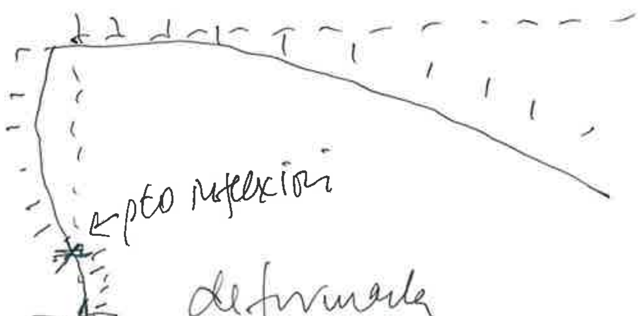

1/2
parabola
1/2
lineare
5/2
lineare
5/2
M

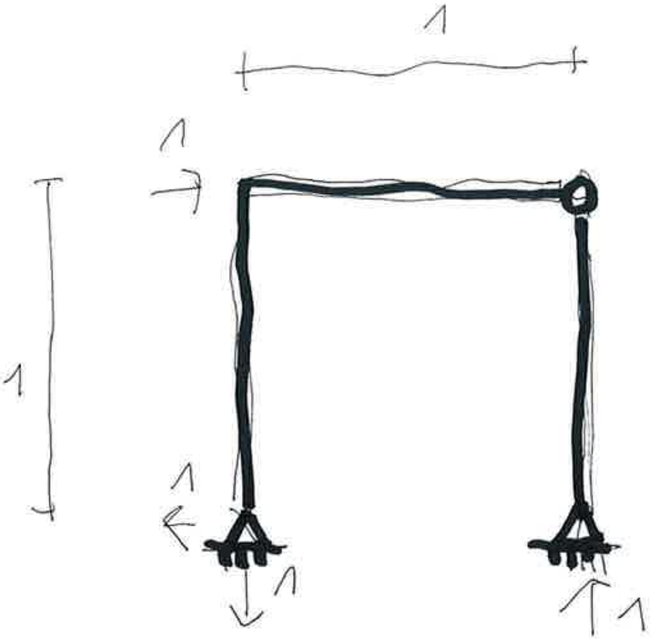

q=1
2
1
2
1
1/2
1
1

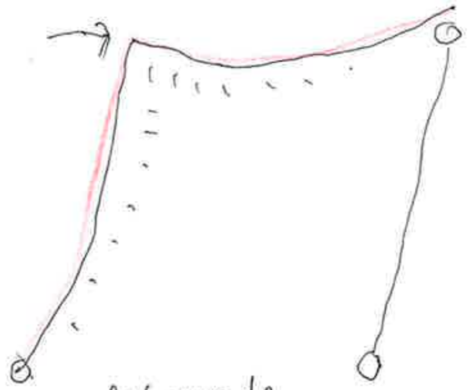

2
1


1
2

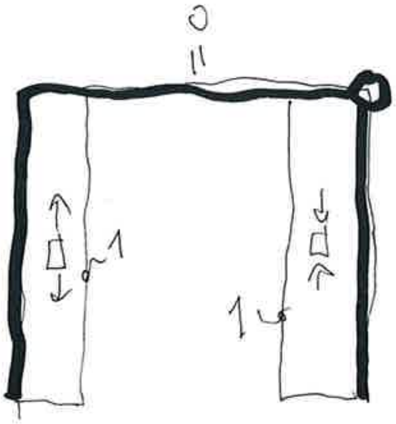

3/2
lineal
1/2
parabola


pto inflexión
deformada

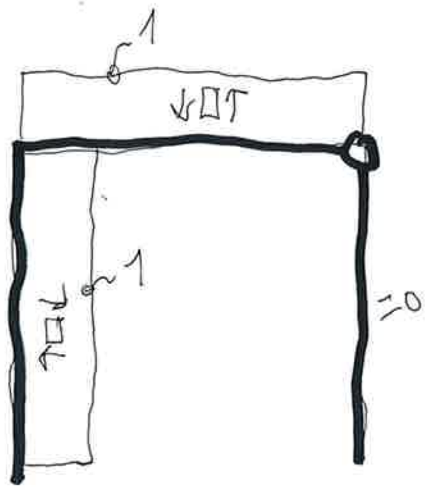




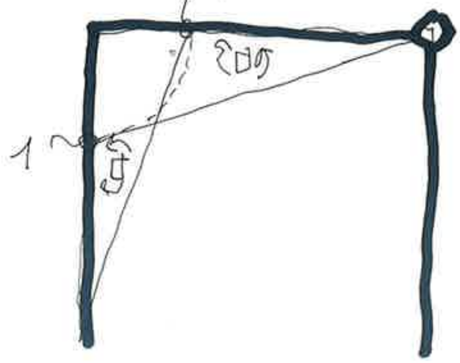
deformada



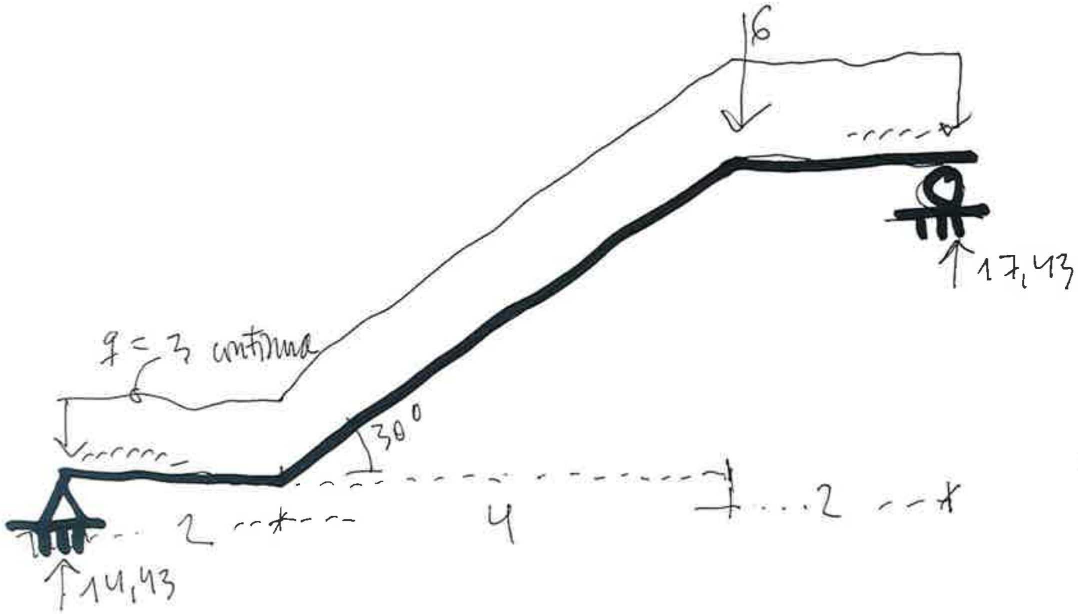
N

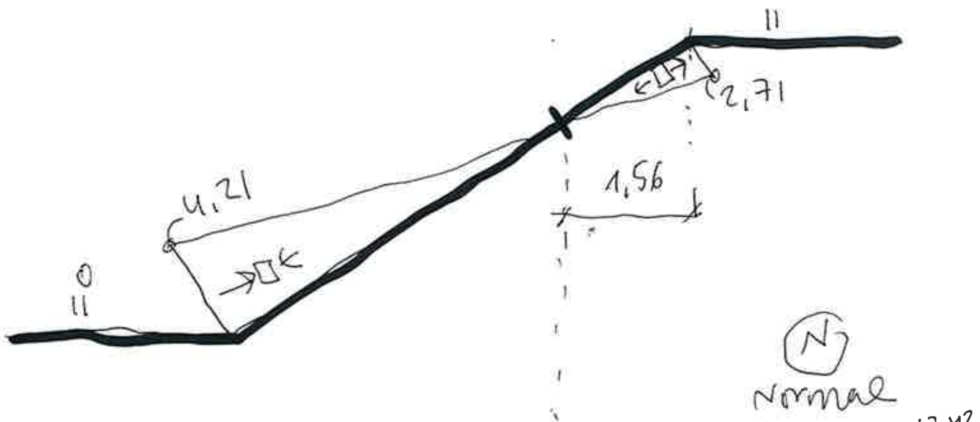


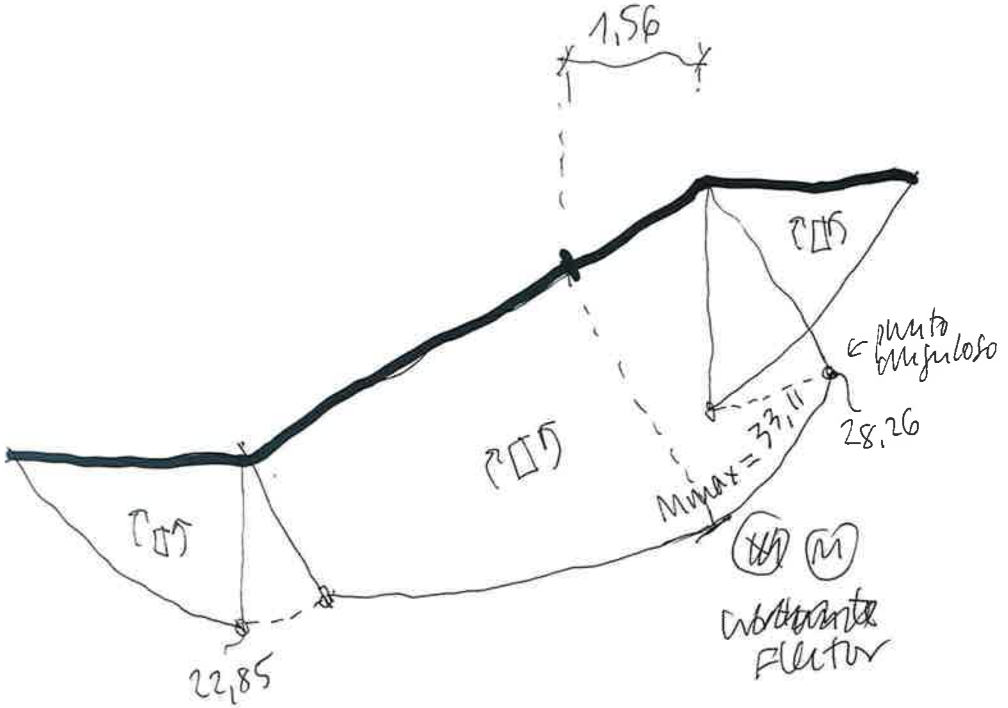
V

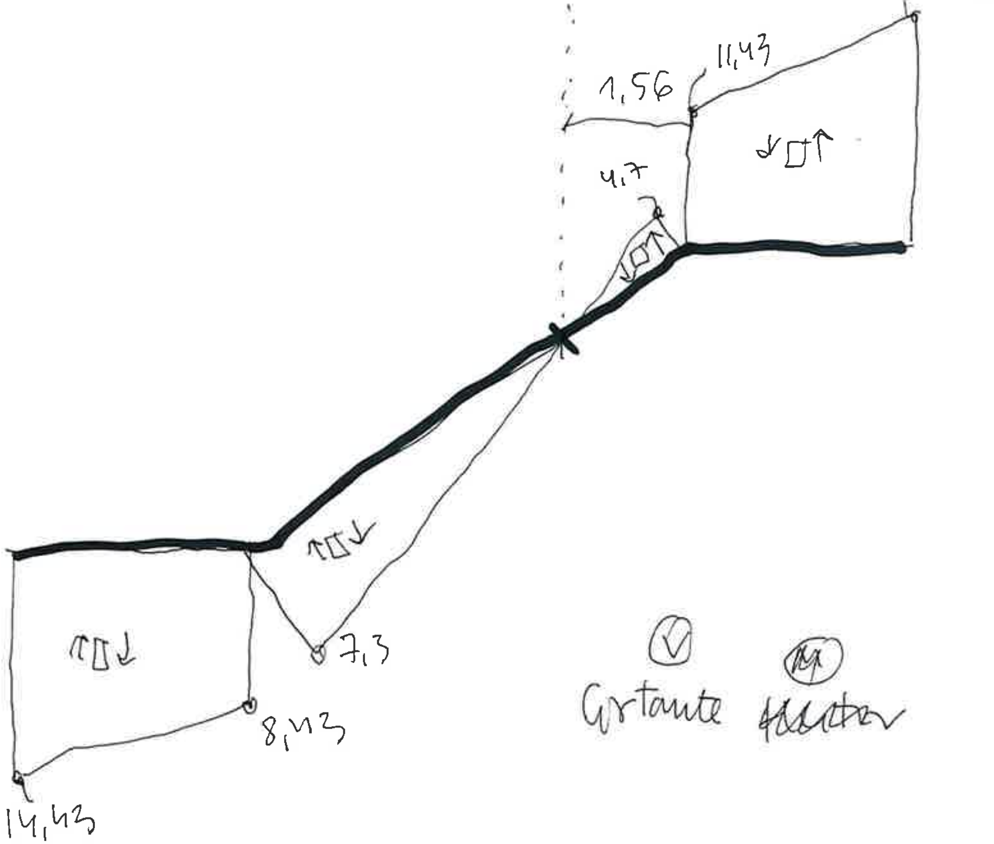


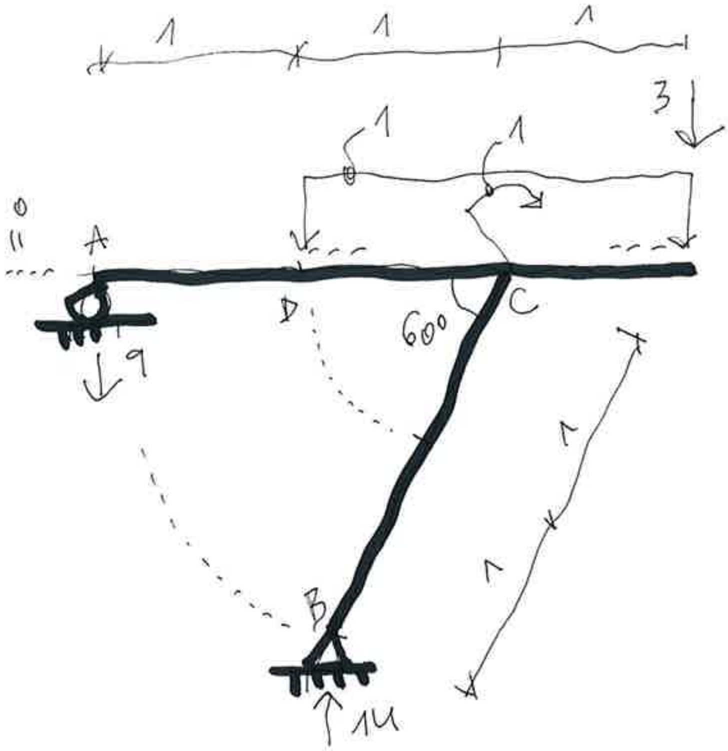
M

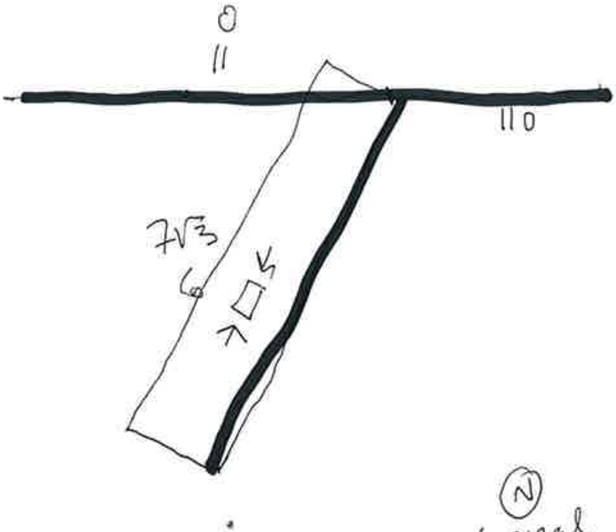

16
q = 3 uniforme
30°
17,43
14,43
2
4
2


0
2,71
1,56
4,21
0
N
Normal

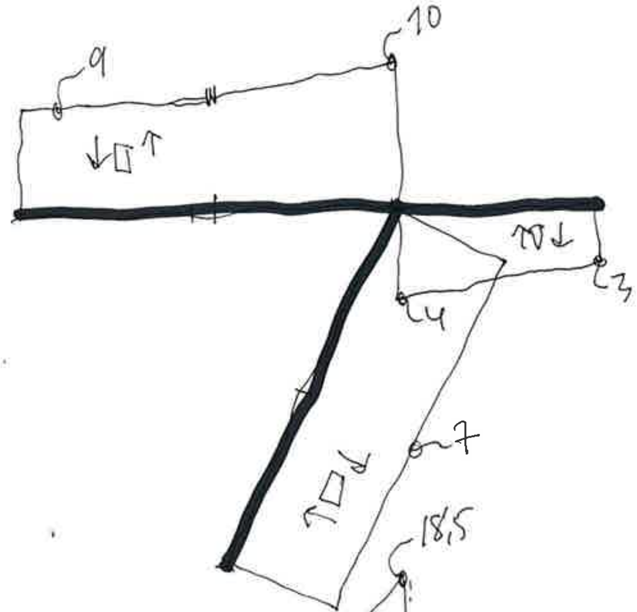

1,56
punto
dudoso
28,26
Mmax = 33,11
22,85
M
Momento
Flector


17,43
11,43
1,56
4,7
7,3
8,43
14,43
V
Cortante

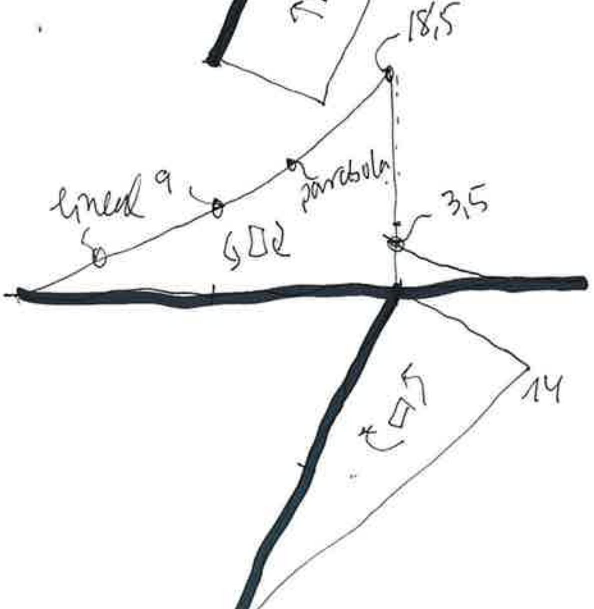




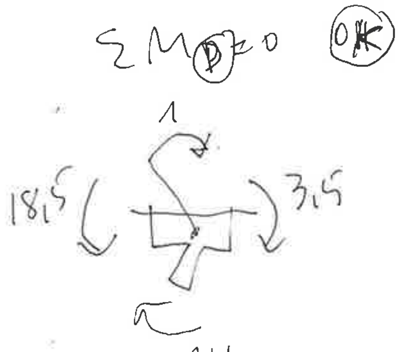
Normal

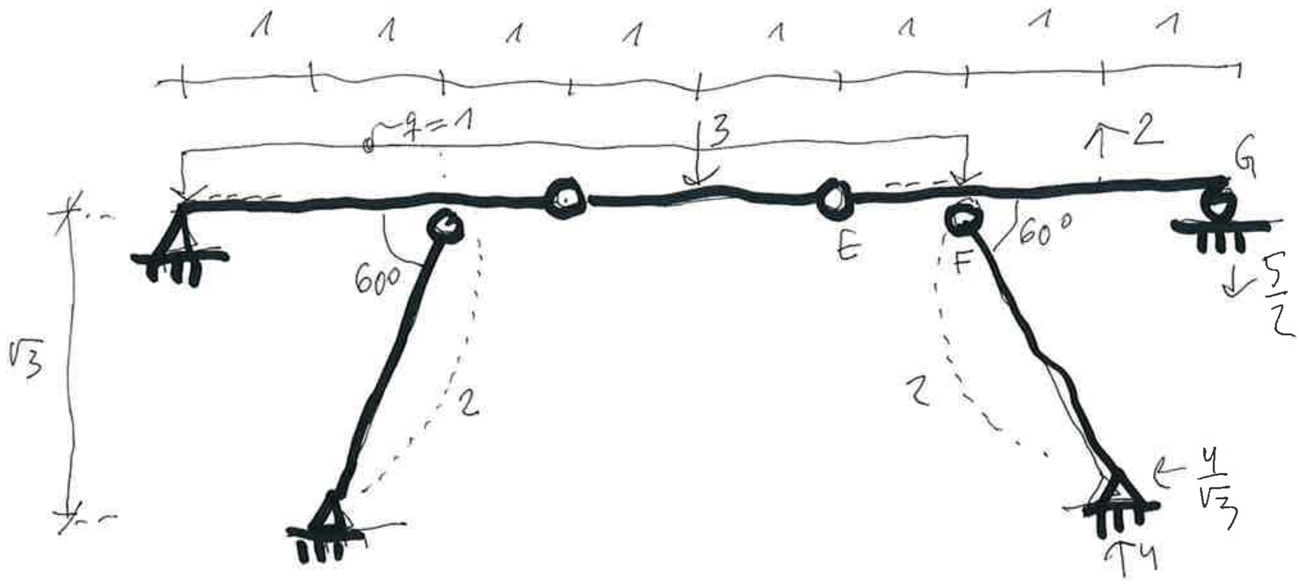


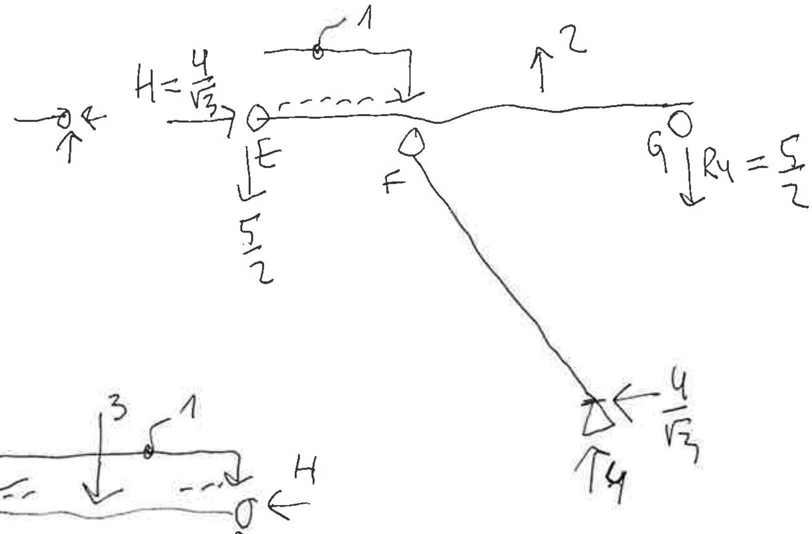
cortante

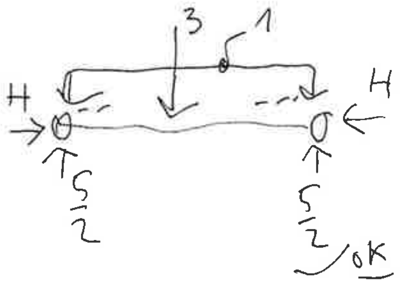


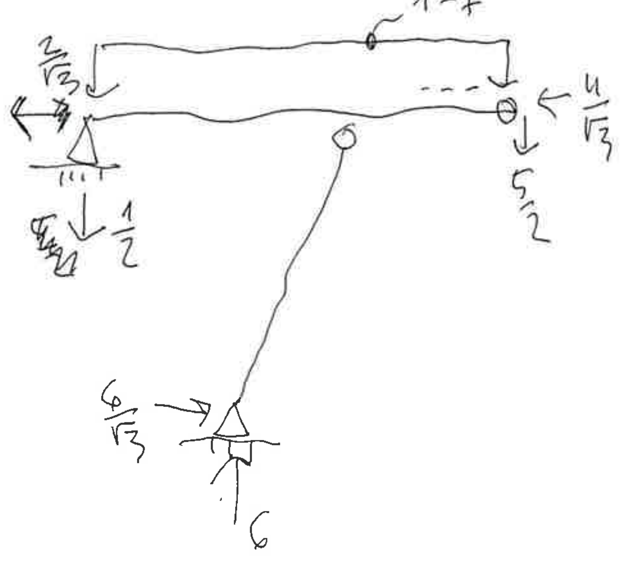
flector

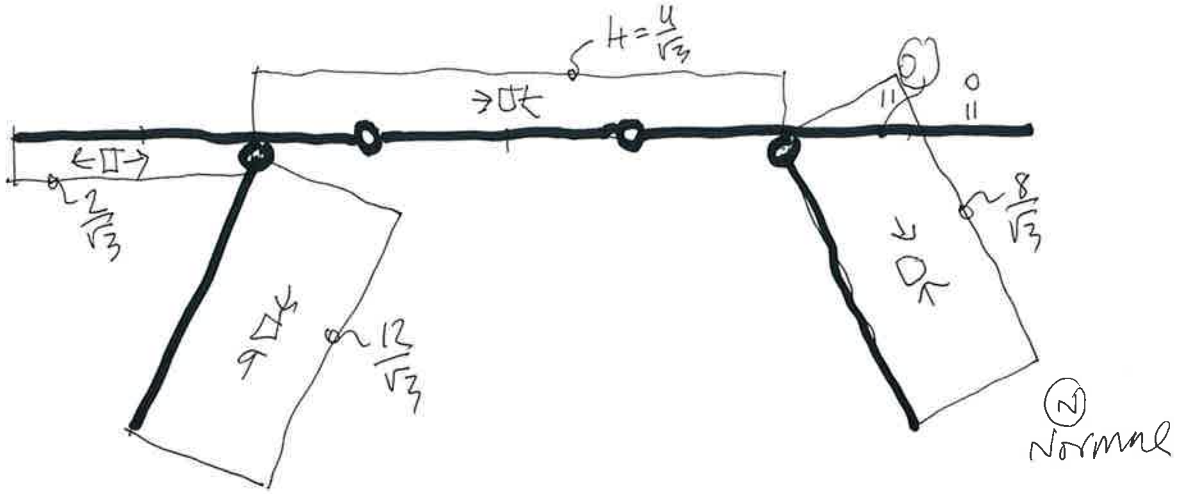



1 1 1 1 1 1 1 1
q = 1
3
2
G
E
F
60°
600
2
2
√3
5/2
4/√3
4

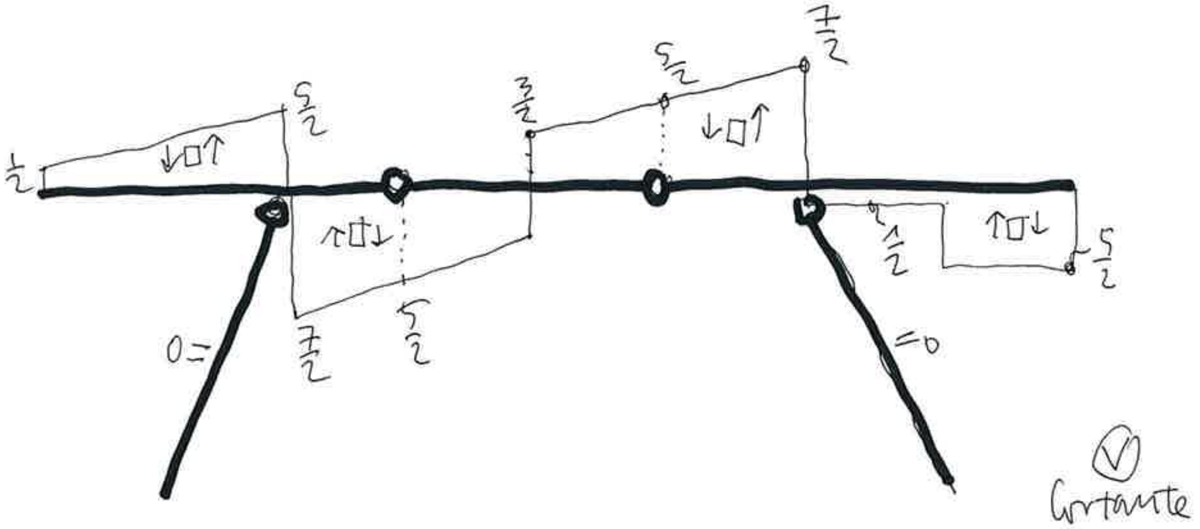

H = 4/√3
1
2
E
F
G
5/2
R4 = 5/2
4/√3
4


3
1
H
H
5/2
5/2
OK

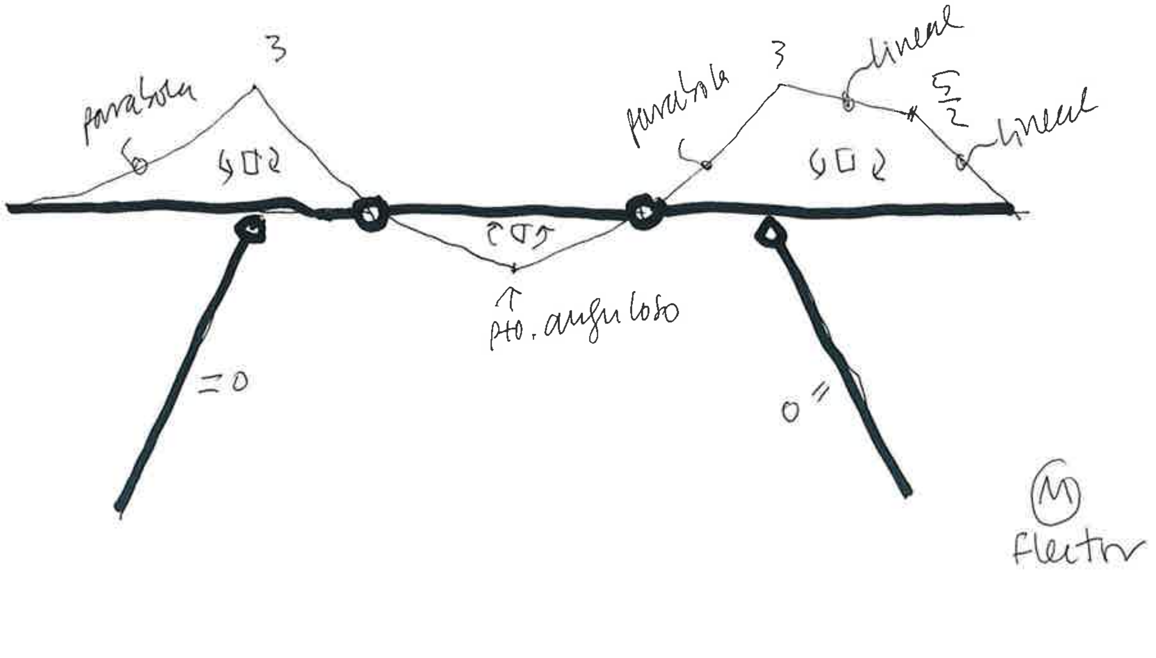

1 = q
2/√3
4/√3
5/2
1/2
6/√3
6



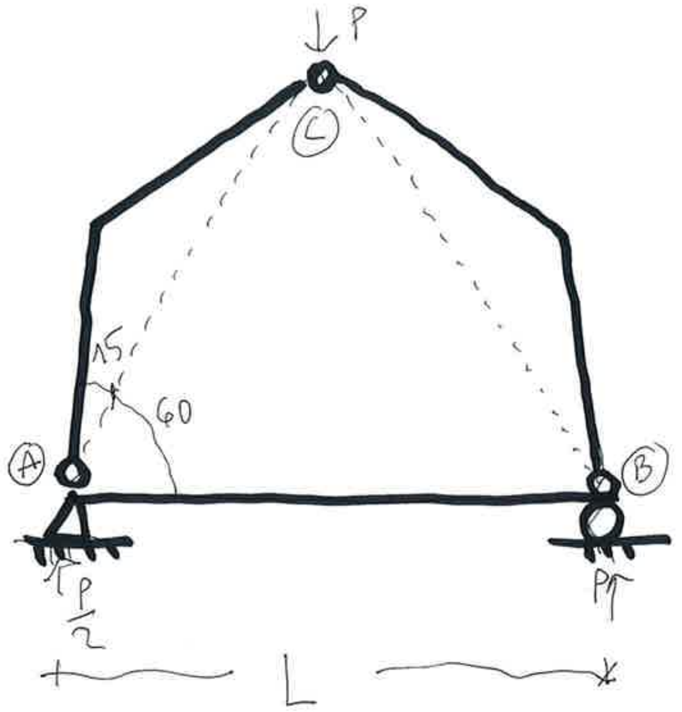
H = 4/√3
2/√3
12/√3
0
8/√3
N
Normal

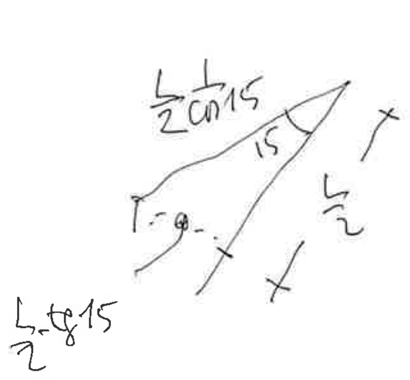


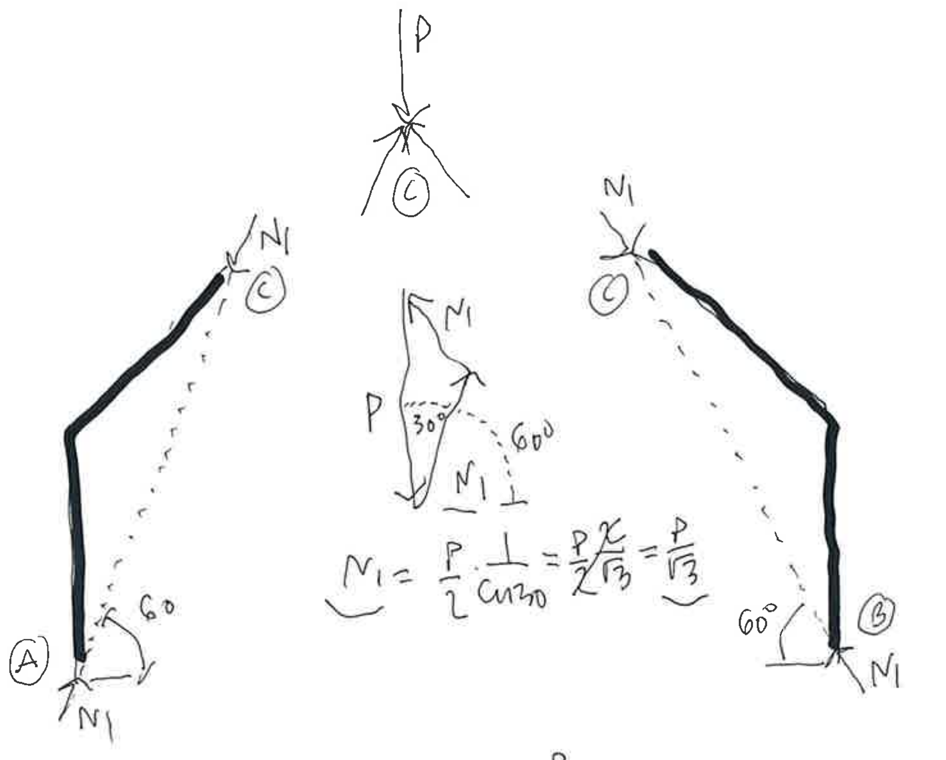
1/2
5/2
3/2
5/2
7/2
7/2
5/2
0
1/2
5/2
0
V
Cortante



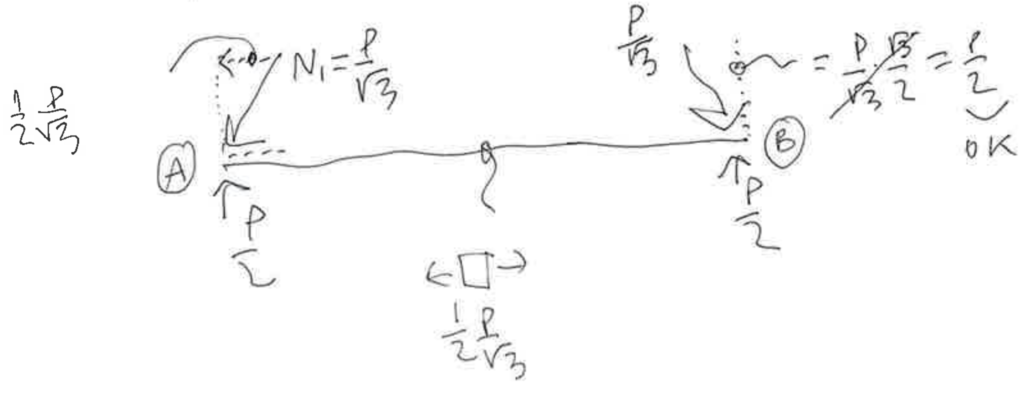
3
parabola
3
parabola
lineal
5/2
lineal
pto. anguloso
0
0
M
Flector

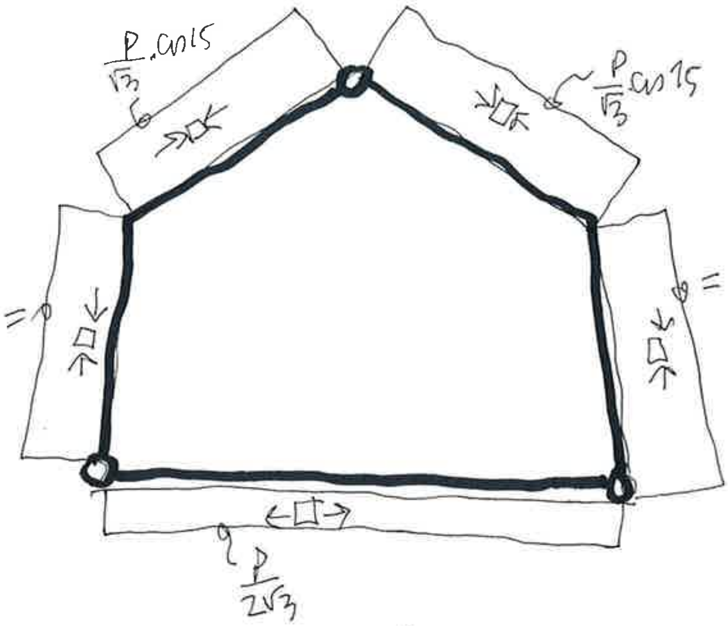




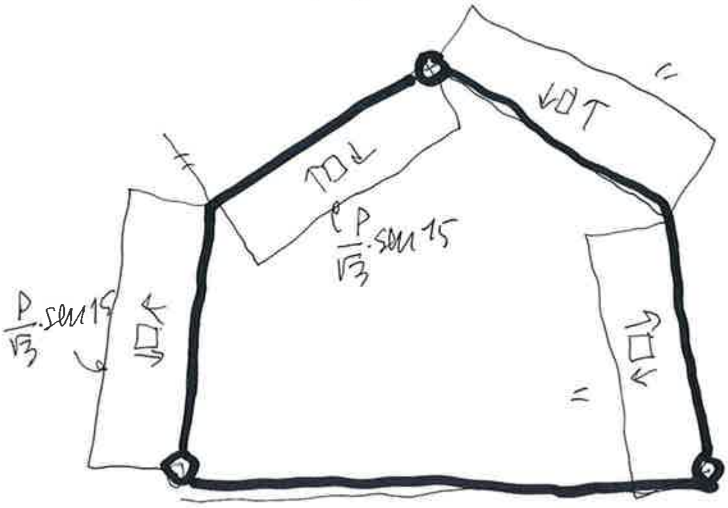


$$N_1 = \frac{P}{2}\cdot\frac{1}{\cos 30} = \frac{P}{\cancel{2}}\frac{\cancel{2}}{\sqrt{3}} = \frac{P}{\sqrt{3}}$$



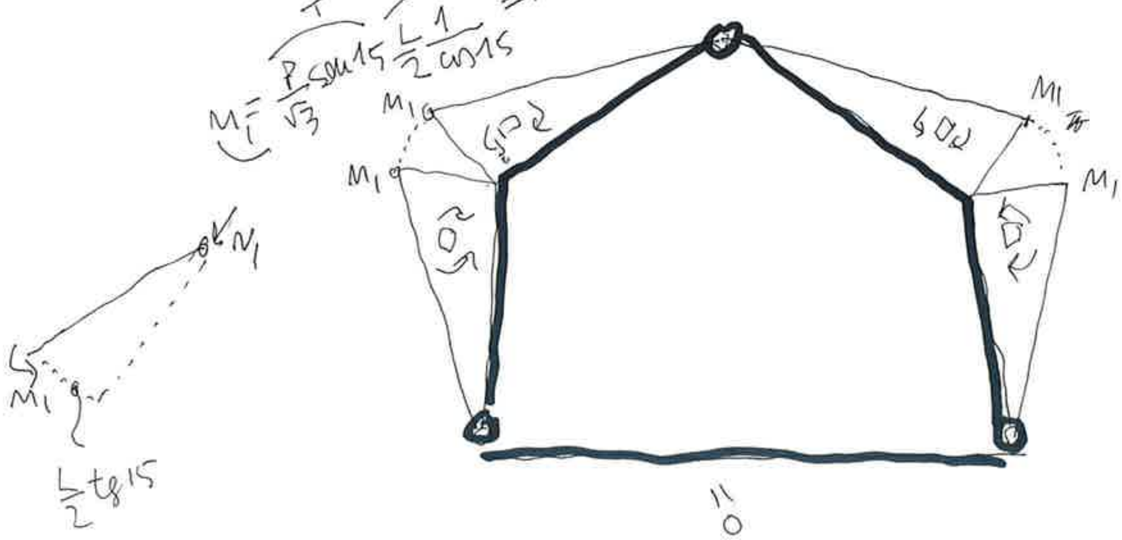


(N) Normal

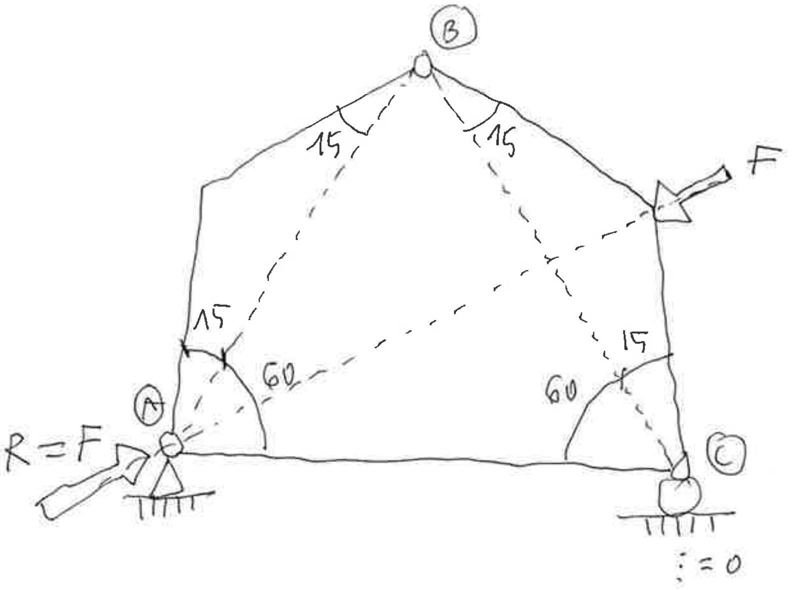


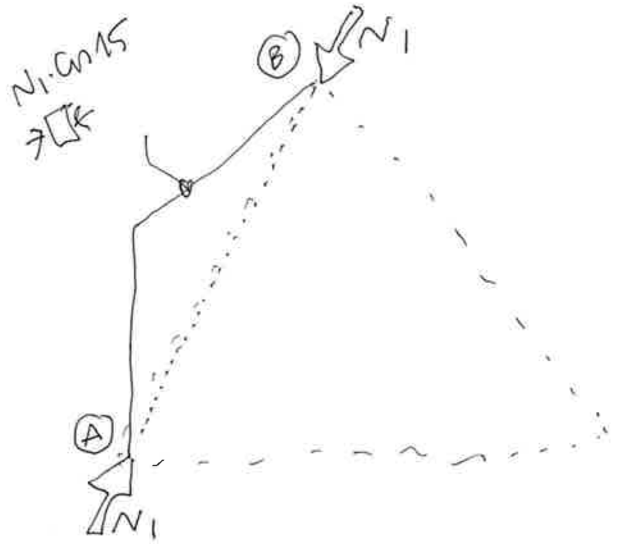
(V) Cortante

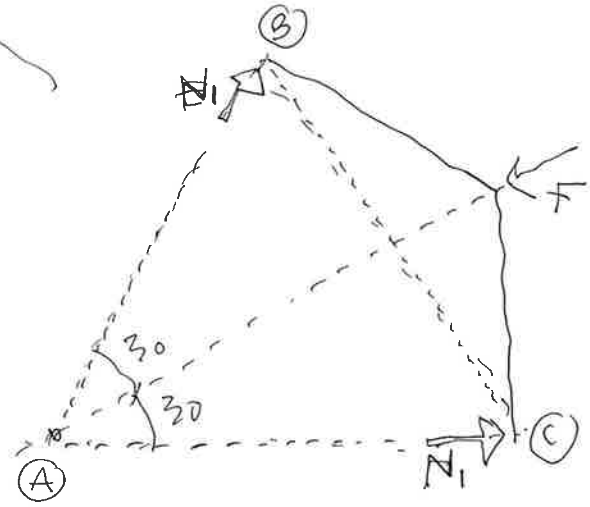
$$M_1 = \overbrace{\frac{P}{\sqrt{3}} \operatorname{sen} 15}^{T} \overbrace{\frac{L}{2} \frac{1}{\cos 15}}^{d} = M_1 \cdot \left(\frac{L}{2} \operatorname{tg} 15\right)$$

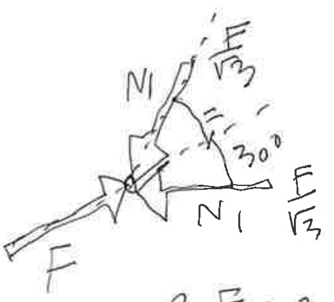


(M) Flector





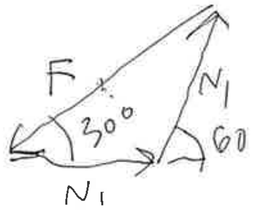




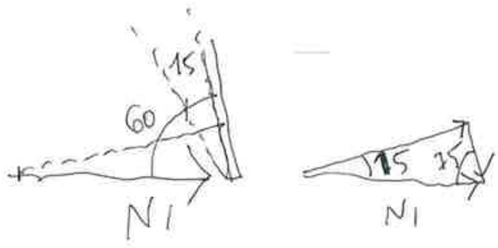
"cierre"

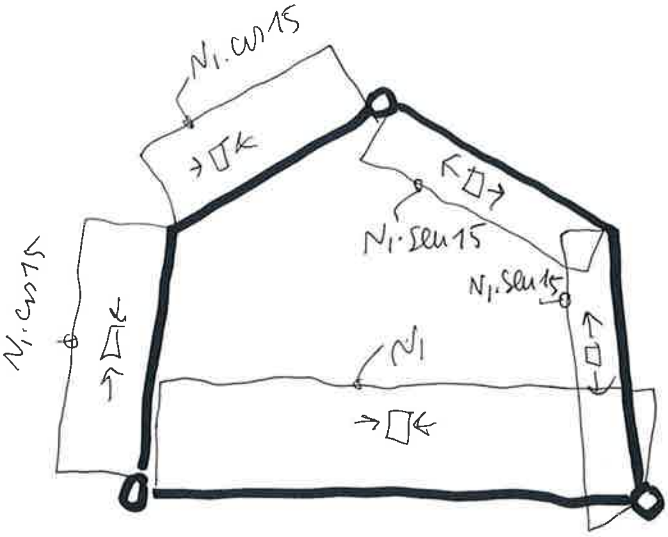
$$2 \cdot \frac{F}{\sqrt{3}} \cdot \cos 30 = 2\frac{F}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2} = \underset{ok}{\underline{\underline{F}}}$$

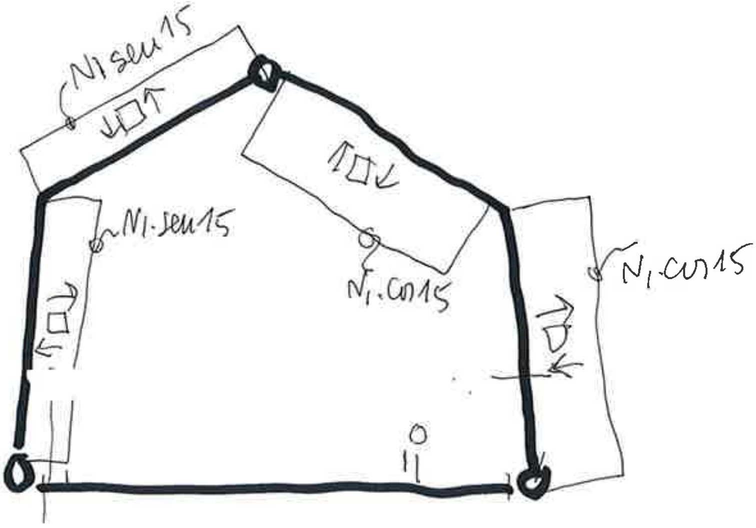
hay equilibrio

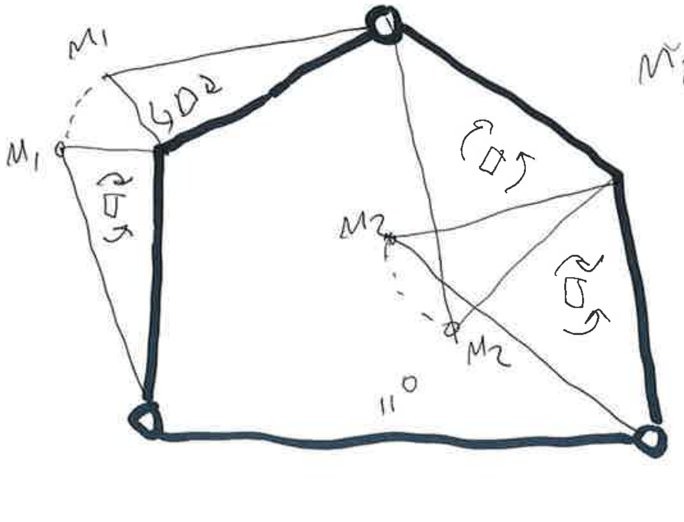


$$\underline{N_1} = \frac{F}{2} \cdot \frac{1}{\cos 30} = \frac{F}{2} \frac{2}{\sqrt{3}} = \underline{\frac{F}{\sqrt{3}}}$$



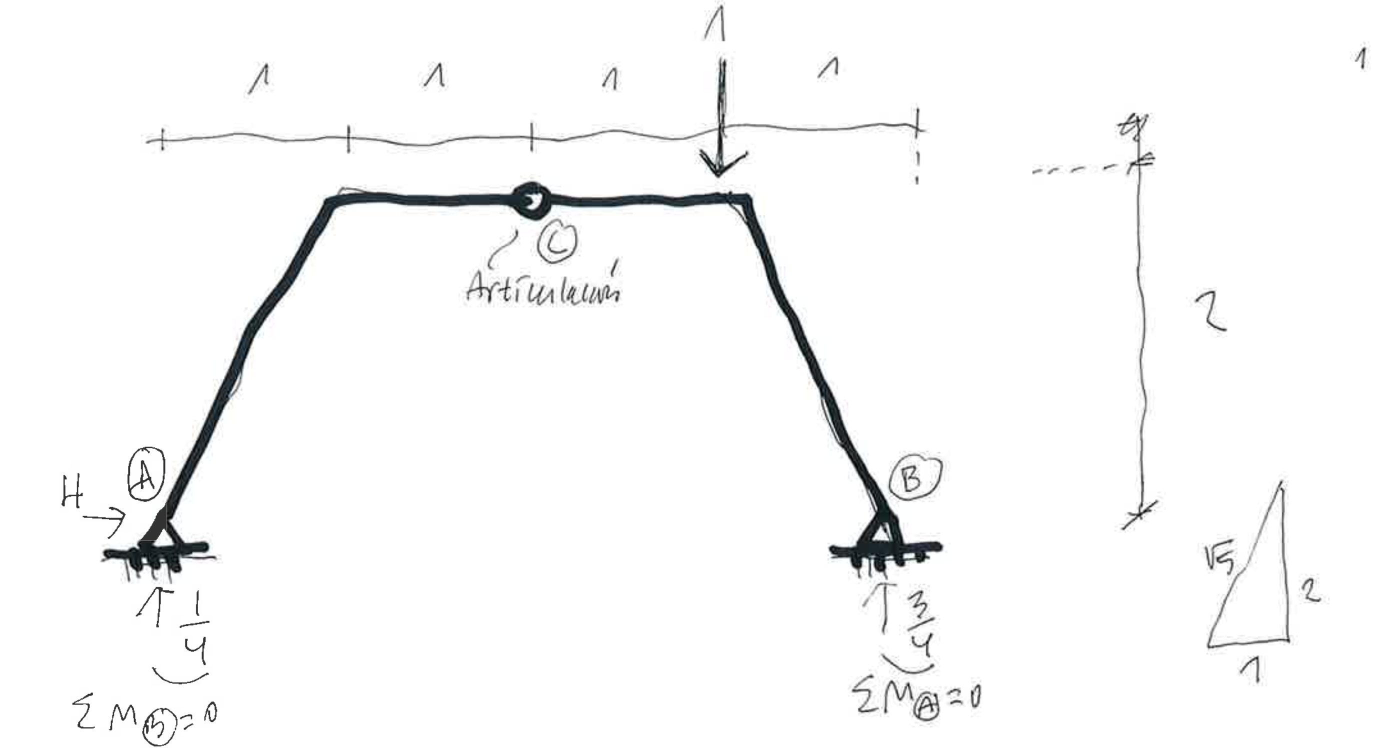


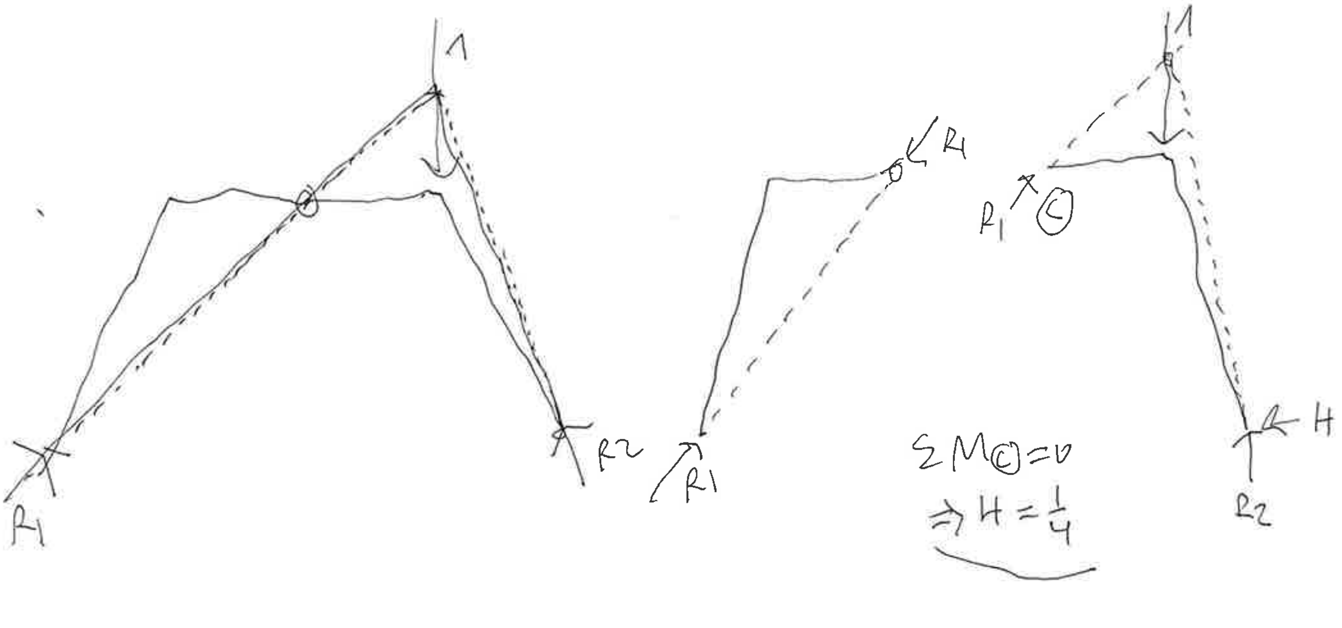


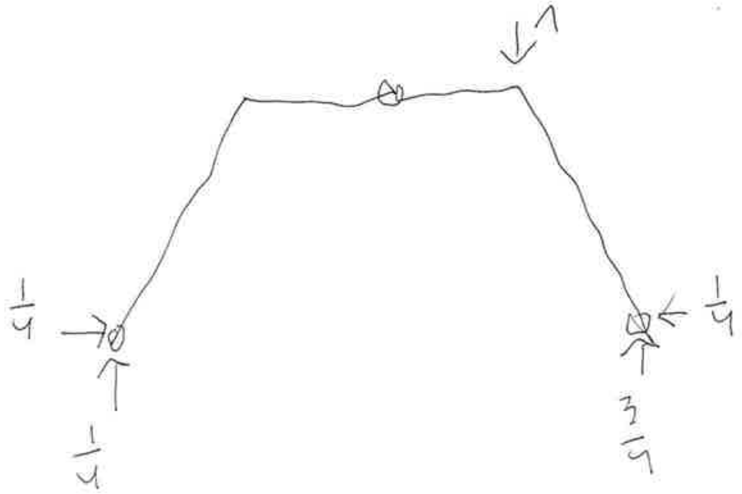


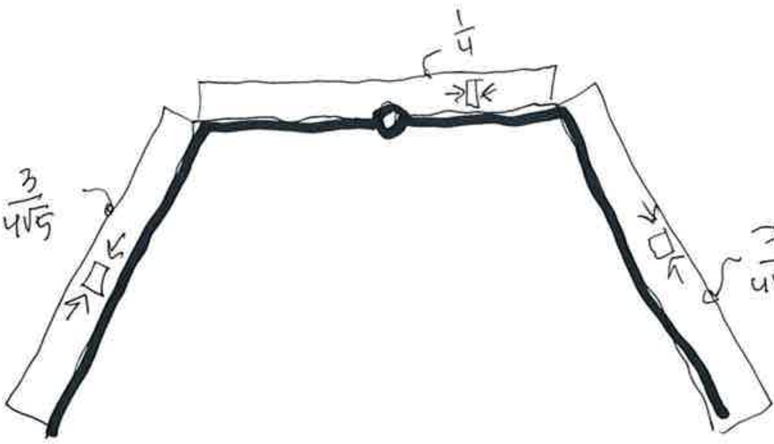
$$M_1 = N_1 \cdot \frac{L}{2} tg15$$

$$M_2 = \overbrace{N_1 \cdot \cos 15}^{cortante} \cdot \overbrace{\frac{L}{2}\frac{1}{\cos 15}}^{d}$$

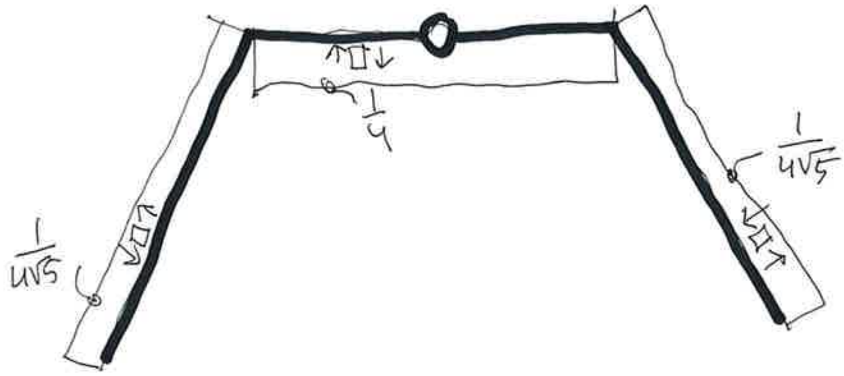

1
1 1 1 1
2
C
Articulación
A
B
H
1/4
3/4
ΣM(B)=0
ΣM(A)=0
√5
2
1


1
R1
R2
R1
R1
C
H
R2
ΣM(C)=0
⇒ H = 1/4

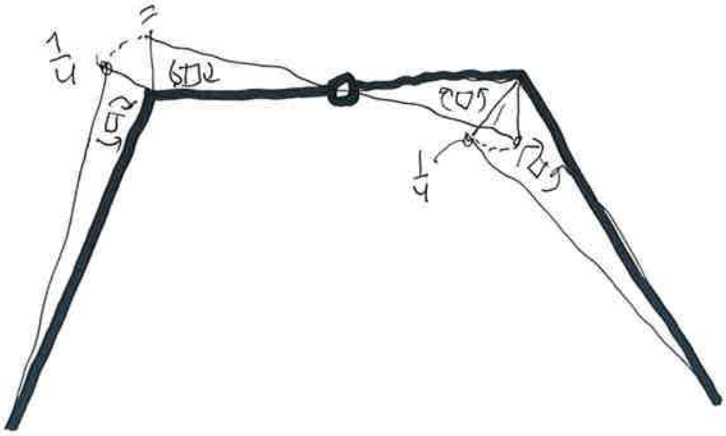

1
1/4
1/4
1/4
3/4



N
Normale



V
Cortante



M
Flettor.